Equations of motion for the transverse and longitudinal modes

In this appendix we outline some details of the steps required to compute the power spectrum of the transverse and longitudinal modes. We start with the background equations of motion.

Taking $\nu = 0$ in Eq. (6), we get

$$\partial^i \dot{A}_i - \partial^k \partial_k A_t + (am)^2 A_t = 0\,, \tag{50}$$

and taking $\nu = i$

$$\begin{aligned}\ddot{A}^i + H\dot{A}^i + m^2 A^i = a^{-2}\left[\partial_k \partial^k A^i - \partial^i \partial^k A_k\right] \\ + \partial^i \left(\dot{A}_t + HA_t\right)\,.\end{aligned} \tag{51}$$

Contracting Eq. (6) with ∂_μ, we obtain an integrability condition which reads

$$(am)^2 \dot{A}_t - m^2 \partial^i A_i + 3H\left(\partial^k \partial_k A_t - \partial^i \dot{A}_i\right) = 0\,. \tag{52}$$

Combining the above equation with Eq. (50), and replacing in Eq. (51), we obtain

$$\ddot{A}^i + H\dot{A}^i + m^2 A^i - a^{-2}\partial^k \partial_k A^i = -2H\partial^i A_t\,. \tag{53}$$

Since inflation homogenizes the vector field, $\partial_i A_\mu = 0$, hence Eq. (50) implies $A_t = 0$, while the spatial components obey Eq. (7).

Now, we perturb around the homogeneous components as in Eq. (8). At first order, the perturbations δA_μ obey the same equations of motion given in Eq. (50) and (53) since they are linear. Let us switch to Fourier space by expanding the perturbations as

$$\delta A_\mu(t,x) \equiv \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \delta\mathscr{A}_\mu(t,k) e^{ik\cdot x}\,. \tag{54}$$

Inserting this in Eq. (50) we get the following constraint

$$\delta\mathscr{A}_t + i\frac{k^j \delta\dot{\mathscr{A}}_j}{k^2 + (am)^2} = 0\,, \tag{55}$$

which allows us to write $\delta\mathscr{A}_t$ in terms of $\delta\mathscr{A}_i$. Using this in Eq. (53) we get

$$\delta\ddot{\mathscr{A}}^i + H\delta\dot{\mathscr{A}}^i + \left[m^2 + \left(\frac{k}{a}\right)^2\right]\delta\mathscr{A}^i + 2Hk^i \frac{k^j \delta\dot{\mathscr{A}}_j}{k^2 + (am)^2} = 0\,. \tag{56}$$

The last step consists in defining the longitudinal and transverse components as

$$\delta\mathscr{A}^i_\parallel \equiv k^i \left(\frac{k^j \delta\mathscr{A}_j}{k^2}\right)\,,\ \delta\mathscr{A}^i_\perp \equiv \delta\mathscr{A}^i - \delta\mathscr{A}^i_\parallel\,, \tag{57}$$

and replacing in the latter equation, we get Eqs. (9) and (10).