Mathematics

Original article

On the Kronecker problem and partially ordered sets with involution

Sobre el problema de Kronecker y conjuntos parcialmente ordenados con involución

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Abstract

We consider the classical Kronecker problem on two linear operators between two finite-dimensional vector spaces and we provide a new short solution using a connection between the matrix version of the Kronecker problem and the matrix problem associated to a partially ordered set with involution.

Keywords: Kronecker problem; Partially ordered set with involution; Indecomposable representation; Matrix problem.

Resumen

Consideramos el clásico problema de Kronecker sobre dos operadores lineales entre dos espacios vectoriales de dimensión finita y presentamos una nueva solución corta usando una conexión entre la versión matricial del problema de Kronecker y el problema matricial asociado a un conjunto ordenado con involución.

Palabras clave: Problema de Kronecker; Conjunto parcialmente ordenado con involución; Representación indescomponible; Problema matricial.

Introduction

The Kronecker problem consists in classifying all pairs of linear transformations between two finite-dimensional vector spaces over a field k. A partial solution was given by Weierstrass for the so called now regular case (**Weierstrass,** 1868), and a complete solution, including both the regular and the singular (non-regular) cases, was given by Kronecker (**Kronecker,** 1890). Over the past decades, numerous solutions have been presented using different approaches: using techniques from linear algebra: (**De Vries,** 1984; **Dieudonné,** 1946; **Gabriel & Roiter,** 1992); using cohomological techniques: (**Benson,** 1995); using categorical and homological methods: (**Auslander *et al.,*** 1997; **Ringel,** 1984). More recently, Zavadskij used a matrix approach to solve the problem and even provided a generalization to the semilinear case, as well as some applications to the representation theory of partially ordered sets with additional structure (**Zavadskij,** 2007); in **Dmytryshyn *et al.*** (2016), the authors provide a generalization to vector spaces and their quotient space and subspace. It is also worth mentioning the applications of the Kronecker problem to systems of linear differential equations (**Gantmacher,** 1959).

Here we propose yet another linear algebra approach to provide a solution to the classification of pairs of linear operators between two vector spaces. Our method is based on

the fact that the matrix version of the Kronecker problem coincides with the corresponding matrix problem for a partially ordered set with involution consisting of two incomparable points. We hope our approach will help to achieve a better understanding of the categories of representations involved.

The structure of this paper is the following: in Section 1 we give some preliminaries about the Kronecker problem and the category of representations of a partially ordered set with involution; we establish the correspondence between the matrix problem for the partially ordered set with involution consisting of two incomparable points and the matrix version of the Kronecker problem. We also prove that the category of representations of a partially ordered set with involution is a Krull-Schmidt category which is not abelian, in general. In Section 2 we obtain the solution to the Kronecker problem by solving a matrix problem corresponding to certain partially ordered set with involution.

The authors are grateful to the referee for their valuable remarks and suggestions. Following their recommendation, we intend to use the reduction procedure to study some other tame problems of Gelfand-type.

1 The Kronecker problem and ordered sets with involution

1.1 The Kronecker problem

Definition 1.1. Given a field k, we will consider quadruples $U = (U_1, U_2, \phi_\alpha, \phi_\beta)$, where U_1 and U_2 are vector spaces of finite dimension over k, and $\phi_\alpha \colon U_1 \to U_2$ and $\phi_\beta \colon U_1 \to U_2$ are linear transformations. For any pair of quadruples $U = (U_1, U_2, \phi_\alpha, \phi_\beta)$ and $V = (V_1, V_2, \psi_\alpha, \psi_\beta)$, their *direct sum* is the quadruple given by

$$U \oplus V = (U_1 \oplus V_1, U_2 \oplus V_2, \phi_\alpha \oplus \psi_\alpha, \phi_\beta \oplus \psi_\beta).$$

We say that two quadruples $U = (U_1, U_2, \phi_\alpha, \phi_\beta)$ and $V = (V_1, V_2, \psi_\alpha, \psi_\beta)$ are *isomorphic* if there exists a pair of k-linear isomorphisms $f_1 \colon U_1 \to V_1$ and $f_2 \colon U_2 \to V_2$ such that the following diagrams commute:

$$
\begin{array}{ccc}
U_1 & \xrightarrow{\phi_\alpha} & U_2 \\
{\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\
V_1 & \xrightarrow{\psi_\alpha} & V_2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
U_1 & \xrightarrow{\phi_\beta} & U_2 \\
{\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\
V_1 & \xrightarrow{\psi_\beta} & V_2
\end{array},
$$

i.e., f_1 and f_2 are bijective k-linear transformations such that

$$\psi_\alpha \circ f_1 = f_2 \circ \phi_\alpha \quad \text{and} \quad \psi_\beta \circ f_1 = f_2 \circ \phi_\beta.$$

In this case, we write $U \simeq V$. We say that a nonzero quadruple U is *indecomposable* if $U \simeq V \oplus W$ implies $V = 0$ or $W = 0$; otherwise, we call U *decomposable*.

The Kronecker problem consists in obtaining the classification of all the indecomposable quadruples, up to isomorphism.

Since we are considering finite-dimensional vector spaces, we can reformulate the Kronecker problem using the matrix language: a quadruple $U = (U_1, U_2, \phi_\alpha, \phi_\beta)$ can be seen, by fixing ordered bases for U_1 and U_2, as a pair of matrices (A, B) of the same size $m \times n$ ($m = \dim_k U_2$ and $n = \dim_k U_1$); such a pair will be called a *matrix presentation* for the Kronecker problem. Two matrix presentations (A', B') and (A, B) are isomorphic if there exist

non singular matrices X, of size $m \times m$, and Y, of size $n \times n$, (which correspond to changing the chosen bases for U_1 and U_2) such that

$$A' = XAY^{-1} \quad \text{and} \quad B' = XBY^{-1}.$$

From this perspective, the Kronecker problem consists in obtaining the classification of the indecomposable matrix presentations (A, B) with respect to transformations of *simultaneous equivalence*; i.e., transformations having the following form:

$$(A, B) \longmapsto (XAY^{-1}, XAY^{-1}),$$

where $X \in GL_m(k)$ and $Y \in GL_n(k)$ (here, for a positive integer q, we denote by $GL_q(k)$ the general linear group of non singular $q \times q$ matrices of elements of k).

1.2 Partially ordered sets with involution and their representations

Definition 1.2. A *partially ordered set with involution* is a triple $(\mathscr{P}, \preccurlyeq, \Theta)$ where $(\mathscr{P}, \preccurlyeq)$ is a partially ordered set and Θ is the set of equivalence classes associated with an equivalence relation $\sim$ on $\mathscr{P}$, such that each equivalence class has at most two elements (an equivalent way to define this type of ordered sets is by considering triples $(\mathscr{P}, \preccurlyeq, *)$ where $(\mathscr{P}, \preccurlyeq)$ is a partially ordered set and $*$ is an *involution on* $\mathscr{P}$; that is, a function $*\colon \mathscr{P} \to \mathscr{P}$ such that $*^2 = 1_{\mathscr{P}}$). Thus, for every $x \in \mathscr{P}$, its equivalence class $[x]$ is either a singleton $[x] = \{x\}$ or has two elements $[x] = \{x, x_1\}$, where $x \sim x_1$. The cardinality of the class $[x]$ is denoted as $r(x)$. If $r(x) = 1$, we say that x is a *small point* and represent it as $\circ$ in the Hasse diagram; if $r(x) = 2$, we say that x is a *large point* and represent it as $\bullet$ in the Hasse diagram.

Remark 1. It is common to refer to the ordered set with involution $(\mathscr{P}, \preccurlyeq, \Theta)$ simply as $(\mathscr{P}, \Theta)$. The equivalence class of a large point $[x] = \{x, x_1\}$ is often identified with the ordered pair (x, x_1), and the equivalence class of a small point $[y] = \{y\}$ is often identified with the element y. If we remove the restriction on the number of elements in the equivalence classes, we obtain *ordered sets with an equivalence relation*, which have been studied in **Bondarenko** and **Zavadskij** (1991) and **Zavadskij** (1991). If every equivalence class is a singleton (i.e., if $* = 1_{\mathscr{P}}$), the ordered set with involution $(\mathscr{P}, \preccurlyeq, \Theta)$ is simply (isomorphic to) the ordinary partially ordered set $(\mathscr{P}, \preccurlyeq)$.

Let's now define the category of representations of an ordered set with involution over a field k. For our purposes, it is sufficient to use the treatment in **Zavadskij** (1991); for a more comprehensive description of the category, see **Cifuentes** (2021, Sections 1.2 and 1.3).

If U_0 is a finite-dimensional k-vector space, for a large point $x \in \mathscr{P}$ with $[x] = (x, x_1)$, we denote by $U_0^{r(x)}$ the direct sum $U_0 \oplus U_0$ of two copies of U_0, where the first summand is indexed by x and the second summand is indexed by x_1:

$$U_0^{r(x)} = U_{0,x} \oplus U_{0,x_1}.$$

If x is a small point, we let $U_0^{r(x)} = U_0$.

For a point $w \in \mathscr{P}$, we denote

$$\pi_w\colon U_0^{r(w)} \to U_{0,w}$$

the canonical projection onto the summand indexed by w, and we call

$$\iota_w\colon U_{0,w} \to U_0^{r(w)}$$

the canonical injection of the summand indexed by w. For $v, w \in \mathscr{P}$, the composite

$$U_0^{r(v)} \xrightarrow{\pi_v} U_{0,v} = U_0 = U_{0,w} \xrightarrow{\iota_w} U_0^{r(w)}$$

will be denoted $\varepsilon_{v,w}$. Therefore, we have

$$\varepsilon_{v,w} = \iota_w \circ \pi_v \colon U_0^{r(v)} \to U_0^{r(w)}.$$

Definition 1.3. Given a field k and a partially ordered set with involution $(\mathscr{P}, \Theta)$, a collection

$$U = (U_0, U_{[x]})_{[x] \in \Theta}$$

is called a *representation of* $(\mathscr{P}, \Theta)$ *over* k if it satisfies the following conditions:

1. U_0 is a finite-dimensional k-vector space.

2. $U_{[x]} \subseteq U_0^{r(x)}$, for all classes $[x] \in \Theta$.

3. $\varepsilon_{x,y}(U_{[x]}) \subseteq U_{[y]}$, if $x \prec y$ in $\mathscr{P}$.

If x is a small point, we write simply U_x instead of $U_{[x]}$. If x is a big point with $[x] = (x, x_1)$, we can also write $U_{(x,x_1)}$ as an alternative to $U_{[x]}$.

Definition 1.4. For a representation over k

$$U = (U_0, U_{[x]})_{[x] \in \Theta}$$

of the poset with involution $(\mathscr{P}, \Theta)$, its *dimension* is the vector

$$d = \underline{\dim} U = (d_0, d_{[x]})_{[x] \in \Theta},$$

where

$$d_0 = \dim U_0 \quad \text{and}$$

$$d_{[w]} = \dim \left(U_{[w]} \Big/ \sum_{\substack{x \prec y \\ y \in [w]}} \varepsilon_{xy}(U_{[x]}) \right),$$

A *morphism* $f \colon U \to V$ between two representations $U = (U_0, U_{[x]})_{[x] \in \Theta}$ and $V = (V_0, V_{[x]})_{[x] \in \Theta}$ of $(\mathscr{P}, \Theta)$ over k is a k-linear transformation $f \colon U_0 \to V_0$ such that

$$f^{|C|}(U_C) \subseteq V_C, \text{ for all } C \in \Theta,$$

where $f^{|C|} \colon U_0^{|C|} \to V_0^{|C|}$ is the map induced by f in a natural way ($|C|$ denotes the cardinality of the set C). A morphism $f \colon U \to V$ is an *isomorphism* if $f \colon U_0 \to V_0$ is an isomorphism of vector spaces such that $f^{|C|}(U_C) = V_C$ for every $C \in \Theta$. If there exists an isomorphism between U and V, we write $U \simeq V$.

It is easy to see that, with the objects and morphisms defined above, we obtain a category, which we denote as $\mathrm{rep}(\mathscr{P}, \Theta, k)$, and which we call the *category of representations of* $(\mathscr{P}, \Theta)$ *over* k.

Definition 1.5. If U and V are representations of $(\mathscr{P}, \Theta)$ over k, the set

$$\mathrm{Hom}(U, V) = \{f \mid f \colon U \to V \text{ is morphism}\}$$

of morphisms from U to V with the usual operations of addition and scalar multiplication by elements of k has the structure of a k-vector space; it is called the *homomorphism space from U to V*. Moreover, the composition of morphisms is bilinear. In particular, the set

$$\mathrm{End}(U) = \mathrm{Hom}(U, U)$$

with the operations of addition, scalar multiplication by elements of k, and composition conforms the *endomorphism algebra of U*. This algebra is a subalgebra of the algebra $\mathrm{End}_k(U_0)$ of k-linear transformations from U_0 to U_0.

The *direct sum* of two representations $U = (U_0, U_{[x]})_{[x] \in \Theta}$ and $V = (V_0, V_{[x]})_{[x] \in \Theta}$ in the category $\mathrm{rep}(\mathscr{P}, \Theta, k)$ is the representation $U \oplus V$ defined as follows:

$$U \oplus V = (U_{[x]} \oplus V_{[x]})_{[x] \in \Theta}.$$

A representation W is *indecomposable* if $W \simeq U \oplus V$ implies $U = 0$ or $V = 0$; otherwise, W is called *decomposable*.

Now we summarize some of the properties of the category $\mathrm{rep}(\mathscr{P}, \Theta, k)$:

Proposition 1.6. *Given a field k and a partially ordered set with involution $(\mathscr{P}, \Theta)$, the following holds:*

(a) *The category $\mathrm{rep}(\mathscr{P}, \Theta, k)$ is additive.*

(b) *$\mathrm{rep}(\mathscr{P}, \Theta, k)$ is a Krull-Schmidt category.*

(c) *The category $\mathrm{rep}(\mathscr{P}, \Theta, k)$ is, in general, **not** abelian.*

Proof. (a) The zero object is the *zero representation* 0 in which all vector spaces are 0 (for small points) or $0 \oplus 0$ (for big points). The direct sum, as defined, can be extended inductively to any finite number of objects, and it can be easily shown to be their biproduct.

(b) Thanks to (a), the category is additive. An inductive argument shows that every non zero object either is indecomposable or decomposes in a finite direct sum of indecomposable objects. It only remains to establish that each indecomposable has a local endomorphism ring. For this, let U be a representation of $(\mathscr{P}, \Theta)$. Since $\mathrm{End}\, U$ is a finite-dimensional k-algebra, it is an Artinian ring and thus, a semiperfect ring. Then, there exists a set $\{e_1, \ldots, e_n\} \subseteq \mathrm{End}\, U$ of orthogonal idempotents such that $\sum_{i=1}^n = 1$ and such that $e_i \mathrm{End}\, U e_i$ is local, for all $i \in \{1, \ldots, n\}$. For an indecomposable U, necessarily $n = 1$ and $\mathrm{End}\, U$ is a local ring.

(c) Let's consider the following partially ordered set with involution, which we will call *dyad with involution*:

$$\mathscr{D} = \left\{ \begin{array}{cc} \bullet & \bullet \\ a & b \end{array} \right\},$$

that is, $\mathscr{D}$ consists of a pair of incomparable points which are equivalent ($\Theta = \{(a,b)\}$). Let's consider the following representations for $\mathscr{D}$ over an arbitrary field k:

$$U = (U_0, U_{(a,b)}) = (k, 0 \oplus k) \quad \text{and} \quad U' = (U'_0, U'_{(a,b)}) = (k, 0 \oplus 0).$$

The identity $1_k \colon k \to k$ is a morphism from U' to U which is a monomorphism: if we consider morphisms $g, h \colon U'' \to U'$ such that $1_k \circ g = 1_k \circ h$, then it is immediate that $g = h$. Analogously one can also see that $1_k \colon k \to k$ is an epimorphism. On the other hand, it is not an isomorphism, since $1_k^2(0 \oplus 0) \neq 0 \oplus k$. Our result follows because $\mathrm{rep}(\mathscr{D}, \Theta, k)$ is not a balanced category (See **Pareigis**, 1970, Lemma 2(c), p. 165). □

1.3 Matrix problems

Representations of partially ordered sets with an equivalence relation were introduced by Nazarova and Roiter in matrix language (**Nazarova** & **Roiter**, 1973). Here we will follow the ideas of **Zavadskij** (1991) for the special case of partially ordered sets with involution.

A *matrix presentation* of a partially ordered set with involution $(\mathscr{P},\Theta)$ over the field k is a matrix M over k divided into vertical blocks M_x, indexed by the elements $x \in \mathscr{P}$, such that $x \sim y$ implies that the number of columns of M_x equals the number of columns of M_y.

Two matrix presentations are called isomorphic if one of them can be turned into the other by applying a finite sequence of the following *admissible transformations*:

AT1. Elementary row transformations of the whole matrix M.
AT2. Simultaneous transformations of columns of the matrices M_x and M_y, if $x \sim y$.
AT3. Additions of columns of M_x to columns of M_y, if $x \prec y$.

For the dyad with involution

$$\mathscr{D} = \left\{ \begin{matrix} \bullet \\ a \end{matrix} \quad\quad \begin{matrix} \bullet \\ b \end{matrix} \right\}$$

the problem then consists in finding canonical forms for matrices of the form

$$M = \boxed{\begin{array}{c|c} A & B \end{array}},$$

where the number of columns of the blocks A and B are equal, under admissible transformations of type AT1 and AT2 (the solid dots below both blocks are used to indicate that the corresponding points are equivalent as well as a reminder that column transformations inside those blocks must be performed simultaneously).

Let's reformulate these ideas in a more precise way, following **Gabriel** and **Roiter** (1992).

Definition 1.7. Given a field k, a *matrix problem of size $r \times s$* is a pair $(G,\mathfrak{M})$ formed by an *underlying set* $\mathfrak{M} \subseteq k^{r \times s}$ and a group $G \subseteq GL_r(k) \times GL_s(k)$ acting on $\mathfrak{M}$ by $(X,Y)M = XMY^{-1}$. The matrix problem is *separated* if $G = G_1 \times G_2$, where $G_1 \subseteq GL_r(k)$ and $G_2 \subseteq GL_r(k)$. The matrix problem $(G,\mathfrak{M})$ is *linear* if G is the group D^i of invertible elements of a subalgebra $D \subseteq k^{r \times r} \times k^{s \times s}$. Solving a matrix problem $(G,\mathfrak{M})$ consists in classifying the orbits of $\mathfrak{M}$ under the action of G.

In other words, a matrix problem of size $r \times s$ is a pair $(\mathfrak{M},G)$ consisting of a set $\mathfrak{M}$ of matrices of size $r \times s$, together with a group G, which corresponds to the admissible row and column transformations of the matrices of $\mathfrak{M}$ which determines an equivalence relation. The goal is then to find a *canonical form*, i.e., to determine a set of canonical matrices such that each G-equivalence class contains exactly one canonical matrix.

In particular, for the dyad with involution

$$\mathscr{D} = \left\{ \begin{matrix} \bullet \\ a \end{matrix} \quad\quad \begin{matrix} \bullet \\ b \end{matrix} \right\}$$

we have the separated linear matrix problem $(G,\mathfrak{M})$ with $\mathfrak{M} = k^{m \times 2n}$ and

$$D = \left\{ (X,Z) \,\middle|\, X \in k^{m \times m} \text{ and } Z = \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}, \text{ with } Y \in k^{n \times n} \right\}.$$

In this case, $G = D^i$ is

$$D^i = \left\{ (X,Z) \,\middle|\, X \in GL_m(k) \text{ and } Z = \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}, \text{ with } Y \in GL_n(k) \right\}.$$

A matrix $M \in \mathfrak{M}$ can be interpreted as a block matrix $M = \boxed{\begin{array}{c|c} A & B \end{array}}$ with $A,B \in k^{n \times n}$. With this interpretation, the action of G on $\mathfrak{M}$ is given by

$$(X,Y)M = XMZ^{-1} = X\boxed{\begin{array}{c|c} A & B \end{array}}\begin{bmatrix} Y^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix} = \boxed{\begin{array}{c|c} XAY^{-1} & XBY^{-1} \end{array}}.$$

But this is nothing more than reducing pairs of matrices of the same size by simultaneous equivalence, as explained in Section 1.1. In other words, the matrix problem associated to the dyad with involution is essentially the same as the matrix version of the Kronecker problem.

Remark 2. From the categorical point of view, quadruples $U = (U_1, U_2, \varphi_\alpha, \varphi_\beta)$ over a field k correspond to representations of the well known *Kronecker quiver*, see, for example, **Auslander *et al.*** (1997)

$$\mathscr{K} = \left\{ 1 \; \underset{\beta}{\overset{\alpha}{\Longrightarrow}} \; 2 \right\}.$$

Morphisms between such representations can be defined by dropping the condition of bijectivity from the definition of isomorphic quadruples (See Definition 1.1). We have then the category $\mathrm{rep}(\mathscr{K}, k)$, of representations of $\mathscr{K}$ over k. Even though the matrix problems for the dyad with involution and for the Kronecker problem are the same, the categories $\mathrm{rep}(\mathscr{D}, \Theta, k)$ and $\mathrm{rep}(\mathscr{K}, k)$ are **not** equivalent. Since the category $\mathrm{rep}(\mathscr{K}, k)$ is abelian (See, for example, **Schiffler** (2014)), having an equivalence between $\mathrm{rep}(\mathscr{K}, k)$ and $\mathrm{rep}(\mathscr{D}, \Theta, k)$, would imply that the latter would also be an abelian category (See **Schubert**, 1972, Proposition 16.2.4, p. 169), which contradicts Proposition 1.6(c).

2 The solution to the Kronecker problem

In this section we will solve the matrix problem corresponding to the dyad with involution. According to Section 1.3, this is tantamount to obtaining the classification of all indecomposable presentations of the Kronecker problem. To describe the indecomposable representations of the dyad with involution, we will follow a similar idea to the one given in **Zavadskij** (2007, Theorem 1), which is actually a variation of the "reduction" mechanism used in **Medina** and **Zavadskij** (2004, Theorem 1) to solve the Four Subspace Problem. First, let us set some notation we will use in the matrix forms.

For an integer $n \geq 1$, the matrices $I_n^\uparrow$, $I_n^\downarrow$ are the $(n+1) \times n$ matrices obtained by adjoining a row of zeroes above, below, the identity matrix I_n; i.e.,

$$I_n^\uparrow = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{(n+1)\times n} \quad \text{and} \quad I_n^\downarrow = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(n+1)\times n}.$$
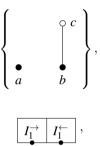
Analogously, the matrices $I_n^\rightarrow$, $I_n^\leftarrow$ are the $n \times (n+1)$ matrices obtained by adjoining a column of zeroes to the right, to the left, of the identity matrix I_n; i.e.,

$$I_n^\rightarrow = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{n\times(n+1)} \quad \text{and} \quad I_n^\leftarrow = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n\times(n+1)}.$$

For $n = 0$, the matrices $I_0^\uparrow$ and $I_0^\downarrow$ are equal and they are "formal" matrices having one row and zero columns, and representing the linear operator $0 \to k$. The matrices $I_0^\rightarrow$ and $I_0^\leftarrow$ are also equal and they are "formal" matrices having zero rows and one column, and representing the linear operator $k \to 0$.

By $F_n(p^s(t))$, for $n \geq 1$, we denote the Frobenius cell of order n corresponding to the minimal polynomial $p^s(t)$, where $p(t)$ is monic and irreducible over k; notice that, in particular, $n = s \cdot \deg p(t)$.

For $n \geq 1$, we denote by $J_n^-(0)$ the Jordan block or order n with eigenvalue 0 and entries 1 below the principal diagonal.

In the matrix presentations we do not draw a block M_x, if all its entries are null. For example, for the partially ordered set with involution

$$\left\{ \begin{matrix} & & \circ\, c \\ & & | \\ \bullet & & \bullet \\ a & & b \end{matrix} \right\},$$

we draw

$$\boxed{I_1^{\rightarrow} \mid I_1^{\leftarrow}}\,,$$

instead of the matrix presentation

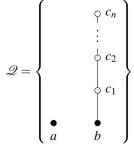$$\boxed{I_1^{\rightarrow} \mid I_1^{\leftarrow} \mid 0}\;.$$

Remark 3. Throughout this section, one could replace all occurrences of $J_n^-(0)$ with $J_n^+(0)$ (the Jordan block or order n with eigenvalue 0 and entries 1 above the principal diagonal). In both representations of type I, one can substitute $J_n^-(0)$ with $J_n^+(0)$ and the resulting matrix forms are equivalent to the original ones. For type V, if one uses $J_n^+(0)$ instead of $J_n^-(0)$, then one must at the same time substitute the rightmost column $\begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$ with $\begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix}^T$.

Now, we present a lemma that will be used in the proof of the main result.

Lemma 2.1. *If U is an indecomposable representation of a partially ordered set with involution $\mathscr{P}$ such that $U_l = U_0$ for some small point $l \in \mathscr{P}$, then the restriction $U' = (U_0, U_i)_{i \in \mathscr{P} \setminus \{l\}}$ to the ordered subset $\mathscr{P} \setminus \{l\}$ is also indecomposable.*

Proof. If we had U' decomposing into a nontrivial direct sum $U' = V \oplus W$, then immediately U would also be decomposable since $U_l = U_0 = V_0 \oplus W_0$. $\qquad\square$

Theorem 2.2. *All the indecomposable representations of the following partially ordered set with involution*

$$\mathscr{Q} = \left\{ \begin{matrix} & & \circ\, c_n \\ & & \vdots \\ & & \circ\, c_2 \\ & & | \\ & & \circ\, c_1 \\ & & | \\ \bullet & & \bullet \\ a & & b \end{matrix} \right\}$$

are exhausted, up to isomorphism, and up to duality, by the following six types of matrix presentations:

$$
0 \quad
\begin{array}{|c|c|}
\hline
a & b \\
\hline
I_n & F_n \\
\hline
\end{array}
\quad , \ \textit{where } F_n = F_n\big(p^s(t)\big) \textit{ and } p(t) \ne t.
$$
$$
n \ge 1
$$

$$
\text{I} \quad
\begin{array}{|c|c|}
\hline
a & b \\
\hline
I_n & J_n^-(0) \\
\hline
\end{array}
\quad \textit{and} \quad
\begin{array}{|c|c|}
\hline
a & b \\
\hline
J_n^-(0) & I_n \\
\hline
\end{array}
\quad .
$$
$$
n \ge 1 \qquad\qquad\qquad n \ge 1
$$

$$
\text{II} \quad
\begin{array}{|c|c|}
\hline
a & b \\
\hline
I_n^{\rightarrow} & I_n^{\leftarrow} \\
\hline
\end{array}
\quad .
$$
$$
n \ge 0
$$

$$
\text{III} \quad
\begin{array}{|c|c|}
\hline
a & b \\
\hline
I_n^{\downarrow} & I_n^{\uparrow} \\
\hline
\end{array}
\quad .
$$
$$
n \ge 0
$$

$$
\text{IV} \quad
\begin{array}{|c|c|c|}
\hline
a & b & c_i \\
\hline
I_n^{\downarrow} & I_n^{\uparrow} & \begin{smallmatrix}1\\0\\ \vdots \\ \vdots \\0\end{smallmatrix} \\
\hline
\end{array}
\quad , \ \textit{for } i \in \{1,\dots,n\}.
$$
$$
n \ge 0
$$

$$
\text{V} \quad
\begin{array}{|c|c|c|}
\hline
a & b & c_i \\
\hline
I_n & J_n^-(0) & \begin{smallmatrix}1\\0\\ \vdots \\ \vdots \\0\end{smallmatrix} \\
\hline
\end{array}
\quad , \ \textit{for } i \in \{1,\dots,n\}.
$$
$$
n \ge 1
$$

Proof. Let $U = (U_0, U_{(a,b)}, U_{c_1}, \dots, U_{c_n})$ be an *indecomposable* representation for $\mathscr{Q}$. We will consider two cases:

(a) There are no points c_i. In this situation we have

$$
\mathscr{Q} = \mathscr{D} = \left\{ \begin{matrix} \bullet \\ a \end{matrix} \qquad \begin{matrix} \bullet \\ b \end{matrix} \right\}
$$

and a matrix presentation for U will have the form

$$
M = \begin{array}{|c|c|}
\hline
A & B \\
\hline
\end{array} \quad .
$$

We will examine two subcases:

Subcase A: the block A is non singular. By applying appropriate admissible operations, we can reduce the block A to an identity block and transform M into the following form:

$$
\begin{array}{|c|c|}
\hline
I_n & B' \\
\hline
\end{array} \quad .
$$

By examining which admissible operations can be applied without altering the identity block, we observe that B' reduces by similarity, and the indecomposability of U

implies that B' can be transformed into the form $F_n = F_n(p^s(t))$ of a unique Frobenius block, resulting in the following form:

$$\boxed{\begin{array}{|c|c|} I_n & F_n \end{array}}\,, \quad n \geq 1.$$

The symmetric situation, in which the block B of the presentation M_U is non-singular, is treated similarly. In this case, we obtain the form:

$$\boxed{\begin{array}{|c|c|} F_n & I_n \end{array}}\,, \quad n \geq 1.$$

Now, if $\det\big(F_n(p^s(t))\big) \neq 0$, then

$$\boxed{\begin{array}{|c|c|} F_n & I_n \end{array}} \simeq \boxed{\begin{array}{|c|c|} I_n & F'_n \end{array}}\,,$$

for some non-singular Frobenius block F'_n. Otherwise (i.e., if $\det\big(Fn(p^s(t))\big) = 0$), we obtain

$$\boxed{\begin{array}{|c|c|} F_n & I_n \end{array}} \simeq \boxed{\begin{array}{|c|c|} J_n^-(0) & I_n \end{array}}\,, \quad n \geq 1,$$
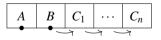
and thus, M_U is of type 0 or I.

Subcase B: one of the blocks A or B is singular. Since we are working up to duality and in the case being considered we can also assume that we are working up to permutations of the points, we can take the block B as being singular by rows. We proceed by induction on $d_0 = \dim U_0$. If $d_0 = 1$, then the dimension vector of U is of the form $(1, d_{(a,b)})$, with $d_{(a,b)} \in \{0, 1, 2\}$. If $d_{(a,b)} = 0$, then M_U is a "formal" matrix with 1 row and 0 columns, corresponding to type III, with $n = 0$. If $d_{(a,b)} = 1$, then $M_U \simeq \boxed{\begin{array}{|c|c|} 1 & 0 \end{array}}$, corresponding to type I, with $n = 1$. The case $d_{(a,b)} = 2$ cannot occur as it would lead to $M_U \simeq \boxed{\begin{array}{|c|c|} 1 & 1 \end{array}}$, which contradicts our assumption of the block B being singular.

Let $d_0 \geq 2$ and suppose the theorem holds for every representation

$$U' = (U'_0, U'_{(a,b)}, U'_{c_1}, \ldots, U'_{c_n}),$$

in which $\dim U'_0 = d'_0$ is such that $d'_0 < d_0$. Consider an indecomposable representation U with $\dim U_0 = d_0$, any of its matrix presentation M_U has the form

$$\boxed{\begin{array}{|c|c|c|c|c|} A & B & C_1 & \cdots & C_n \end{array}}$$

By applying suitable row operations, all linearly dependent rows within the block B become zero and we move the remaining linearly independent rows to the bottom of the matrix, obtaining the following form (from now on, the unmarked blocks in the matrices correspond to zero blocks):

$$\boxed{\begin{array}{|c|c|} A_1 & \\ \hline A_2 & B_2 \end{array}}\,.$$

The upper block A_1 cannot have zero rows, otherwise, U would have trivial direct summands, contradicting its indecomposability. Therefore, we can assume that the rows of A_1 are linearly independent. By applying admissible row operations, we bring A_1 to its row echelon form (by the previous comment, this row echelon form does not
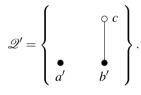
contain zero rows), and maybe using some column exchanges, we get an identity block
in the upper-right corner in the stripe corresponding to the point a. Then by using row
operations, we nullify the block below this identity, producing the matrix form

$$
\begin{array}{|c|c|c|c|}
\hline
 & I_r & & \\
\hline
A' & & B' & C \\
\hline
\end{array}
\quad , \quad r \geq 1,
\tag{2.1}
$$

where we have separated the bottom right block into two sub-blocks $\boxed{B' \mid C}$, with
the condition that the number of columns in B' matches the number of columns in A'.

Admissible transformations which do not change the upper part of the whole matrix
presentation, include column additions from B' to C and the same additions must be
performed from the corresponding columns of A' to the zero block to its right. But
this block can be turned again to zero, with suitable row transformations involving
the identity block above it. Elementary column transformations of C, are allowed,
and when these operations transform the identity block, its form can be recovered by
applying the corresponding inverse row transformations. Therefore the matrix problem
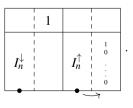to solve in the blocks

$$
\begin{array}{|c|c|c|}
\hline
A' & B' & C \\
\hline
\end{array}
$$

coincides with the matrix problem of the following ordered set with involution

$$
\mathscr{Q}' = \left\{
\begin{array}{cc}
 & \circ\, c \\
 & | \\
 & \\
\bullet & \bullet \\
a' & b'
\end{array}
\right\}.
$$

This matrix presentation corresponds to an indecomposable representation, since oth-
erwise, replacing a non-trivial direct sum decomposition for this presentation in the
matrix (2.1), we would obtain a non-trivial direct sum decomposition for M_U, which
contradicts its indecomposability. Moreover, it satisfies $d_0' < d_0$ (since $r \geq 1$), which
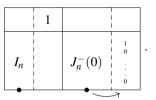means, by the induction hypothesis, it has one of the matrix forms 0 to V.

We then just need to replace in the matrix (2.1) the blocks $\boxed{A' \mid B' \mid C}$ with matrices
of the forms 0 to V, and verify that the resulting matrix again has one of the types 0
to V. This process is straightforward and we describe it in detail below: the types 0
to III have zeroes in the block C, so the matrix (2.1) would have direct summands of
the form $\boxed{1 \mid 0}$, and the hypothesis of indecomposability implies that $M_U \simeq \boxed{1 \mid 0}$,
which is of type I with $n = 1$. If the blocks $\boxed{A' \mid B' \mid C}$ are of type IV, then $r = 1$,
and the matrix (2.1) would have the form

$$
\begin{array}{|c|c|c|}
\hline
 & 1 & \\
\hline
I_n^{\downarrow} & I_n^{\uparrow} & \begin{smallmatrix}1\\0\\ \vdots \\0\end{smallmatrix} \\
\hline
\end{array} \; .
$$

With a convenient permutation, the rightmost column of each vertical stripe become
the first one, and the matrix turns into the form

$$
\begin{array}{|c|c|}
\hline
I_{n+1}^{\downarrow} & I_{n+1}^{\uparrow} \\
\hline
\end{array}
$$

which is of type III. Finally, if the blocks $\boxed{A' \mid B' \mid C}$ are of type V, then $r = 1$, and the matrix (2.1) would have the form:

$$
\begin{array}{|c|c|c|}
\hline
 & 1 & \\
\hline
I_n & J_n^-(0) & \begin{smallmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \\
\hline
\end{array} \; .
$$

Once again, with the permutation that sends the column i to the column $i+1$, and the column $n+1$ to the first one, we obtain:

$$
\boxed{I_{n+1} \mid J_{n+1}^-(0)}
$$

which is of type I.

(b) Let us now assume that there exist $n \geq 1$ points $c_1, c_2, \ldots, c_n$. We will proceed by induction on

$$
m = \sigma(d) = d_0 + d_{(a,b)} + d_{c_1} + d_{c_2} + \cdots + d_{c_n},
$$

where

$$
d_0 = \dim(U_0),
$$
$$
d_{(a,b)} = \dim(U_{(a,b)}),
$$
$$
d_{c_1} = \dim\big(U_{c_1}/U_b\big),
$$
$$
d_{c_2} = \dim\big(U_{c_2}/U_{c_1}\big),
$$
$$
\vdots
$$
$$
d_{c_n} = \dim\big(U_{c_n}/U_{c_{n-1}}\big).
$$

We will consider two subcases:

Subcase A: let's assume that $U_{c_j} = U_0$ for some $j \in \{1, \ldots, n\}$. Since it is impossible to have $m = 1$ according to our assumptions, we will start the induction with $m = 2$. In this case, it is necessary $d_0 = 1$ and $d_{c_j} = 1$ for a unique $j \in \{1, \ldots, n\}$, and all other coordinates of the dimension vector are zero. This corresponds to a representation such that $U_{c_j} = U_0 \simeq k$ and $U_{(a,b)} = U_{c_\ell} = 0$, for $\ell \neq j$, its matrix presentation is type IV with $n = 0$.

Let us consider the case $m \geq 3$. We assume the result holds for any representation of dimension d' with $\sigma(d') < m$. Consider an indecomposable representation U with $\sigma(d) = m$ and set

$$
t = \min\{\, j \mid U_{c_j} = U_0 \,\}.
$$

By repeatedly applying Lemma 2.1, the restriction U', of U to $\mathcal{Q} \setminus \{c_n, c_{n-1}, \ldots, c_{t-1}\}$, is also indecomposable. Since $d_t = \dim(U_t/U_{t-1}) > 0$, we have $\sigma(d') < \sigma(d)$, and the induction hypothesis implies that U' belongs to one of the types 0 to V. In types IV and V, we have $U_{c_i} = U_0$ for all $i \in \{1, \ldots, n\}$, which implies $U_{t-1} = U_0$. We would have $d_t = \dim(U_t/U_{t-1}) = 0$, which is absurd. Therefore, we must have $n = 1$, $\mathcal{Q} \setminus \{c_1\} = \mathcal{D}$, and thus U' can only be of types 0, I, II, or III.

In the following matrix forms:

$$
\begin{array}{cc}
a & b \\
\boxed{\begin{array}{c|c} I_n & F_n \end{array}} & \\
n \geq 1 &
\end{array} \;, \text{ where } F_n = F_n\big(p^s(t)\big) \text{ y } p(t) \neq t,
$$

and

$$\begin{array}{|c|c|} \hline a & b \\ \hline J_n^-(0) & I_n \\ \hline \end{array}$$
$$n \geq 1$$

the block F_n, in the first case, and the block I_n, in the second case, allow us to nullify all the entries in the c_1 block by using appropriate column additions. Therefore, among the types 0 and I, the only possible case remaining is that U' has the form

$$\begin{array}{|c|c|} \hline a & b \\ \hline I_n & J_n^-(0) \\ \hline \end{array},$$
$$n \geq 1$$

then U will have the following form:

$$\begin{array}{|c|c|c|} \hline a & b & c_1 \\ \hline & & \begin{matrix}1\\0\\ \vdots \\ \vdots \\ 0\end{matrix} \\ I_n & J_n^-(0) & \\ \hline \end{array},$$

which corresponds to type V (any other column of c_1 can be annihilated with suitable column operations using the $J_n^-(0)$ block).

Moreover, U' cannot be of type II either, because once again, when considering U, we could eliminate all the elements in the c_1 block by adding the columns of the $I_n^{\leftarrow}$ block. It only remains to examine what happens when U' is of type III:
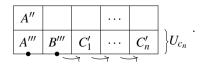
$$\begin{array}{|c|c|} \hline a & b \\ \hline I_n^{\downarrow} & I_n^{\uparrow} \\ \hline \end{array}$$
$$n \geq 0$$.

Upon "reconstructing" U, we would obtain the following matrix

$$\begin{array}{|c|c|c|} \hline a & b & c_1 \\ \hline & & \begin{matrix}1\\0\\ \vdots \\ \vdots \\ 0\end{matrix} \\ I_n^{\downarrow} & I_n^{\uparrow} & \\ \hline \end{array},$$
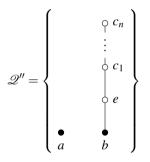
which corresponds to type IV (any other column of c_1 can be annihilated with appropriate column operations using the $I_n^{\uparrow}$ block).

Subcase B: let's now assume that for every $j \in \{1,\ldots,n\}$, we have $U_{c_j} \neq U_0$. For the base case of the induction with $m = 1$, we have $d_0 = 1$ and all other coordinates of the dimension vector must be zero, so U is of type III with $n = 0$. Now, let's consider the case for $m \geq 2$. We assume that the result holds for representations with $\sigma(d') < m$, and we consider an indecomposable representation U with $\sigma(d) = m$. Under our assumption, we have in particular that $U_{c_n} \neq U_0$, and we can place the rows corresponding to U_{c_n} at the bottom of M_U, obtaining a matrix presentation of the form:

$$\left.\begin{array}{|c|c|c|c|c|} \hline A'' & & & \cdots & \\ \hline A''' & B''' & C_1' & \cdots & C_n' \\ \hline \end{array}\right\}U_{c_n} .$$

Similarly to what we did at the beginning of this proof, we observe that the upper block A'' does not have any zero rows (otherwise, we would have trivial direct summands for U, contradicting its indecomposability), and we can assume that the rows of A'' are linearly independent. By applying admissible row operations, we can bring A'' to its row echelon form (as mentioned earlier, this row echelon form does not contain any zero rows) and maybe using some column exchanges we can obtain an identity block. Furthermore, using appropriate row operations, we can eliminate the elements below this identity block, resulting in a matrix of the following form:

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
 & I_s & & & \cdots & \\
\hline
A' & & B' & E & C'_1 \cdots & C'_n \\
\hline
\end{array}
\quad , \quad \text{with } s \geq 1. \tag{2.2}
$$

In this matrix, the block B' has the same number of columns as the block A'. The lower horizontal stripe of this matrix corresponds to a matrix presentation of an indecomposable representation (if it decomposed non-trivially into a direct sum, by placing those summands in the matrix (2.2), this would imply that M_U can also be written as a non-trivial direct sum, which is absurd) of the following ordered set with involution:

$$
\mathscr{Q}'' = \left\{
\begin{array}{cc}
 & c_n \\
 & \vdots \\
 & c_1 \\
 & e \\
a & b
\end{array}
\right\}
$$

This indecomposable representation also satisfies $\sigma(d') < \sigma(d)$ (since $s \geq 1$), and due to our construction, the block E, which admits the same transformations as a small point, is a zero block. Otherwise, we would be in a situation analogous to the one treated above in subcase A, in which the chain of small points has at least two points and the subspace associated with one of them coincides with the main space of the representation, which contradicts our assumption. Thus, U has s direct summands of the form $\boxed{\begin{array}{c|c} 1 & 0 \end{array}}$ and its indecomposability implies $s = 1$ and $M_U \simeq \boxed{\begin{array}{c|c} 1 & 0 \end{array}}$, which is of type I, with $n = 1$. This finishes our proof. $\square$

Corollary 2.3. *All the indecomposable representations of the dyad with involution*

$$
\mathscr{D} = \left\{
\begin{array}{cc}
\bullet & \bullet \\
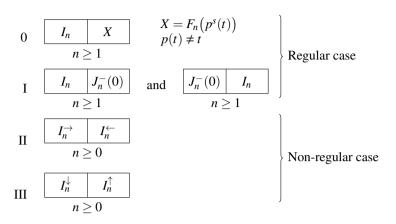1 & 2
\end{array}
\right\}
$$

are exhausted, up to isomorphism, by the presentations of the four types shown in Table 1.

Proof. The result follows immediately from the considerations of the case (a) in the proof of Theorem 2.2. $\square$

Corollary 2.4. *All the indecomposable matrix presentations of the Kronecker problem are exhausted, up to isomorphism, by the presentations of the four types shown in Table 1.*

Proof. The result follows from Corollary 2.3 and the remarks at the beginning of this section. $\square$

Table 1. Indecomposable representations for the Kronecker quiver. Types 0 and I are autodual ($0 = 0^*$ and $I = I^*$), while for types II and III it is the case that $II^* = III$ and $III^* = II$.

Conclusions

By using an elementary matrix based approach related to partially ordered sets with involution, we obtained another solution to the classical Kronecker problem of classifying pair of linear operators between a pair of finite-dimensional spaces over a field k.

The problems we solved in Theorem 2.2, Corollaries 2.3 and 2.4 are of tame representation type, in the sense of **Simson** (1992, Section 14.4); in fact, they are of finite growth type.

In the process of obtaining our solution, we found two non equivalent categories of representations with the same associated matrix problem. In the case of partially ordered sets, matrix problems appeared earlier than the corresponding vector space or categorical approach and they are often a very valuable tool to describe objects and their isomorphism classes. But matrices do not conform a category, actually they do not involve information about morphisms.

The authors aim to research about the precise relation between the categories $\text{rep}(\mathscr{D}, \Theta, k)$ and $\text{rep}(\mathscr{K}, k)$, and to translate that relation to some combinatorial objects such as the corresponding Auslander-Reiten quivers.

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Authors' contribution

ID and GM: both authors actively participated in the conception and development of the project; they worked together to elaborate and review the manuscript.

Conflict of interest

None to declare.

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