Original article

Mathematics

Characteristic-Dependent Linear Rank Inequalities in 21 Variables

Desigualdades Rango Lineales Dependientes de la Característica en 21 variables

Victor Peña-Macias*, Humberto Sarria-Zapata

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional de Colombia, Bogotá, Colombia

Abstract

In Linear Algebra over finite fields, a characteristic-dependent linear rank inequality is a linear inequality that holds by ranks of spans of vector subspaces of a finite dimensional vector space over a finite field of determined characteristic, and does not in general hold over fields with other characteristic. This paper shows a preliminary result in the production of these inequalities. We produce three new inequalities in 21 variables using as guide a particular binary matrix, with entries in a finite field, whose rank is 8, with characteristic 2; 9 with characteristic 3; or 10 with characteristic neither 2 nor 3. The first inequality is true over fields whose characteristic is 2; the second inequality is true over fields whose characteristic is 2 or 3; the third inequality is true over fields whose characteristic is neither 2 nor 3.

Key words: Entropy; Linear rank inequality; Binary matrix; Direct sum in vector spaces.

Resumen

En Álgebra Lineal sobre cuerpos finitos, una desigualdad rango lineal dependiente de la característica es una desigualdad lineal que es válida para dimensiones de sumas de subspacios vectoriales de un espacio vectorial de dimensión finita sobre un cuerpo finito de determinada característica, y no es válida en general sobre cualquier cuerpo de otra característica. Este documento presenta un resultado preliminar referente a la producción de estas desigualdades. Nosotros producimos tres desigualdades nuevas en 21 variables usando como guía una matriz binaria particular, con entradas en un cuerpo finito, cuyo rango es 8, 9 o 10 dependiendo de que la característica sea 2, 3 o distinta de 2 y 3. La primera desigualdad es válida sobre cuerpos de característica 2; la segunda es válida sobre cuerpos de característica 2 o 3; la tercera es válida sobre cuerpos de característica distinta de 2 y 3.

Palabras clave: Entropía; Desigualdad rango lineal; Matriz binaria; Suma directa de espacios vectoriales.

Introduction

In Linear Algebra over finite fields, a linear rank inequality is a linear inequality that is always satisfied by ranks (dimensions) of subspaces of a vector space over any field. Information inequalities are a sub-class of linear rank inequalities (**A. Shen, et al.,** 2000). The Ingleton inequality is an example of a linear rank inequality which is not information inequality (**Ingleton,** 1969), other inequalities have been presented in (**Kinser,** 2011) among others. A characteristic-dependent linear rank inequality is like a linear rank inequality but this is always satisfied by vector spaces over fields of certain characteristic and does not in general hold over fields with other characteristic. In Information Theory, especially in linear network coding, all these inequalities are useful to calculate the linear rates of communication networks (**Dougherty, et al.,** 2013). It is remarkable that the linear rate of a network depends on the characteristic of the scalar field associated to the vector space of the network codes (**Dougherty, et al.,** 2005; **Dougherty, et al.,** 2013). Therefore, when we study linear rates over specific fields, characteristic-dependent lin-

ear rank inequalities are more useful than usual linear rank inequalities.

Characteristic-dependent linear rank inequalities have been presented in (**Blasiak, et al.,** 2011; **Dougherty, et al.,** 2013; **Freiling,** 2014). The technique used by Dougherty et al. to produce these inequalities used as a guide the network flow of some matroidal networks to obtain restrictions over linear solubility; these restrictions imply the inequalities. This technique has produced many inequalities (**Freiling,** 2014), and this is different from the technique used by Blasiak et al. that directly produces two inequalities from the dependency relations of the Fano and non-Fano matroids; it has only produced two inequalities. So we ask ourselves, can more inequalities be produced from other suitable representable matroids and following the ideas of Blasiak et al.?

In this paper, we answer affirmatively. Following a particular case, we show a method to produce characteristic-

dependent linear rank inequalities using as guide a suitable binary matrix; we use the dependency relationships of its columns which are naturally associated with matroid representations. The rank of the desired matrix is 8 if the entries are in a field whose characteristic is 2; the rank is 9 if the characteristic is 3; and the rank is 10 if the characteristic is neither 2 nor 3. We "convert" this property in three inequalities: the first inequality is true over fields whose characteristic is 2; the second inequality is true over fields whose characteristic is 2 or 3; the third inequality is true over fields whose characteristic is neither 2 nor 3. The inequalities do not in general hold over fields with other characteristic. We hope that the techniques presented in this paper can be applied to other types of matrices whose rank behaves in a similar way to the described matrix.

The paper is organized as follows. We introduce some mathematical tools of information theory. After, we show the theorem that produces the described inequalities; before presenting the proof, we give some propositions and lemmas that will be helpful. Finally, we show the proof and some conclusions.

Entropy and inequalities in Linear Algebra

In the following, we introduce the necessary concepts to understand this paper. Let A, B, A_1, ..., A_n be vector subspaces of a finite dimensional vector space V over a finite field $\mathbb{F}$. Let $\sum_{i \in I} A_i$ be the span of A_i, $i \in I$. There is a correspondence between inequalities satisfied by dimensions of spans of vector spaces and inequalities satisfied by entropies of certain class of random variables induced by vector spaces (**A. Shen, *et al.*,** 2000, Theorem 2). We explain that: let f be chosen uniformly at random from the set of linear functions from V to $\mathbb{F}$. For A_1, ..., A_n define the random variables $X_1 = f \mid_{A_1}$, ..., $X_n = f \mid_{A_n}$. For $I \subseteq [n] := \{1, \ldots, n\}$, we have

$$\mathrm{H}\left(X_i : i \in I\right) = (\log |\mathbb{F}|) \dim \left(\sum_{i \in I} A_i\right).$$

The random variables X_1, ..., X_n are called *linear random variables over* $\mathbb{F}$. For simplicity, we identify the entropy of linear random variables with the dimension of the associated subspaces, i.e. $\mathrm{H}(A_i : i \in I)$, the entropy of A_i, $i \in I$, is $\dim\left(\sum_{i \in I} A_i\right)$. With this notation, the *mutual information* of A and B is given by $\mathrm{I}(A; B) = \dim(A \cap B)$. The *codimension of A in V* is given by $\mathrm{codim}_V(A) = \dim(V) - \dim(A)$. We have $\mathrm{H}(A \mid B) = \mathrm{codim}_A(A \cap B)$. In a similar way conditional mutual information is expressed.

We give the following definition in order to fix ideas about inequalities.

Definition 1. Let P be a proper subset of primes, and let I_1, ..., I_k be subsets of $[n]$. Let $\alpha_i \in \mathbb{R}$, for $1 \leq i \leq k$. A

linear inequality of the form

$$\sum_{i=1}^k \alpha_i \mathrm{H}\left(X_j : j \in I_i\right) \geq 0$$

- is called a *characteristic-dependent linear rank inequality* if it holds for all jointly distributed linear random variables X_1, ..., X_n finite fields with characteristic in P, and does not in general hold over fields with other characteristic.
- is called a *linear rank inequality* if it holds for all jointly distributed linear random variables over all finite field.
- is called an *information inequality* if the inequality holds for all jointly distributed random variables.

By definition of linear random variables, we note any information inequality is an inequality which is also satisfied by dimensions of spans of vector spaces. The following inequality is the first linear rank inequality which is not information inequality.

Example 2. (**Ingleton,** 1969) For any A_1, A_2, A_3, A_4 subspaces of a finite dimensional vector space,

$$\mathrm{I}(A_1; A_2) \leq \mathrm{I}(A_1; A_2 \mid A_3) + \mathrm{I}(A_1; A_2 \mid A_4) + \mathrm{I}(A_3; A_4).$$

We can think a characteristic-dependent linear rank inequality like a linear rank inequality that is true over some fields.

We say that $A + B$ is a direct sum, denoted by $A \oplus B$, if $A \cap B = O$. In case that $V = A \oplus B$, the members of this sum are called (mutually) complementary subspaces in V. Alternatively, A_1, ..., A_n are mutually complementary subspaces in V if every vector of V has an unique representation as a sum of elements of A_1, ..., A_n. In this case, π_I denotes the I-projection function $V \twoheadrightarrow \bigoplus_{i \in I} A_i$ given by

$$x = \sum_{i=1}^n x_i \mapsto \sum_{i \in I} x_i.$$

The inequalities of the following lemmas, that we will use later, are valid for linear random variables that hold some additional conditions.

We remark that we use the following notation of intervals: $[j, k] := \{i \in \mathbb{N} : j \leq i \leq k\}$, $[k] = [1, k]$. The sum $A_j + \cdots + A_k$ is denoted by $A_{[j,k]}$, and $A_0 := A_\emptyset := O$.

Lemma 3. *For any subspaces A_1, ..., A_n, A_1', ..., A_n' of a finite dimensional vector space V such that $A_i' \leq A_i$, we have*

$$\mathrm{H}\left(A_{[n]} \mid A_{[n]}'\right) \leq \sum_{i \in [n]} \mathrm{H}(A_i \mid A_i'),$$

with equality if and only if $A_{i+1} \cap A_{[i]} = A_{i+1}' \cap A_{[i]}'$ for all i.

Proof. By induction over n. In case $n = 2$, we have

$$\mathrm{H}\left(A_{[2]} \mid A_{[2]}'\right) = \mathrm{H}\left(A_{[2]}\right) - \mathrm{H}\left(A_{[2]}'\right)$$

$$= \mathrm{H}(A_1) + \mathrm{H}(A_2 \mid A_1) - \mathrm{H}(A_1') - \mathrm{H}(A_2' \mid A_1')$$

$$= \mathrm{H}(A_1 \mid A_1') + \mathrm{H}(A_2) - \mathrm{H}(A_2') - \mathrm{I}(A_1; A_2) + \mathrm{I}(A_1'; A_2')$$

$$= \mathrm{H}\left(A_1 \mid A_1'\right) + \mathrm{H}\left(A_2 \mid A_2'\right) + \left(\mathrm{I}\left(A_1'; A_2'\right) - \mathrm{I}\left(A_1; A_2\right)\right)$$

$$\leq \mathrm{H}\left(A_1 \mid A_1'\right) + \mathrm{H}\left(A_2 \mid A_2'\right) \quad \text{[because } A_i' \leq A_i\text{].}$$

The equality holds if and only if $\mathrm{I}\left(A_1; A_2\right) = \mathrm{I}\left(A_1'; A_2'\right)$. In other words, $A_1 \cap A_2 = A_1' \cap A_2'$ because $A_i' \leq A_i$. Now, we suppose the case $n - 1$ is true. We have

$$\mathrm{H}\left(A_{[n]} \mid A_{[n]}'\right) = \mathrm{H}\left(A_{[n-1]\cup n} \mid A_{[n-1]\cup n}'\right)$$

$$\leq \mathrm{H}\left(A_{[n-1]} \mid A_{[n-1]}'\right) + \mathrm{H}\left(A_n \mid A_n'\right) \quad \text{[from case } n = 2\text{]}$$

$$\leq \sum_{i\in[n]} \mathrm{H}\left(A_i \mid A_i'\right) \quad \text{[from case } n - 1\text{].}$$

The equality holds if and only if $\mathrm{I}\left(A_{i+1}; A_{[i]}\right) = \mathrm{I}\left(A_{i+1}'; A_{[i]}'\right)$. Since $A_i' \leq A_i$ for all i, we have $A_{i+1} \cap A_{[i]} = A_{i+1}' \cap A_{[i]}'$. □

Lemma 4. *For any subspaces A, B, C of a finite dimensional vector space V such that $B \leq A$, we have*

$$\mathrm{H}\left(A \cap C \mid B \cap C\right) \leq \mathrm{H}\left(A \mid B\right),$$

with equality if and only $A + C = B + C$.

Proof. We have

$$\mathrm{H}\left(A \cap C \mid B \cap C\right) = \mathrm{H}\left(A \cap C\right) - \mathrm{H}\left(B \cap C\right)$$

$$= \mathrm{I}\left(A; C\right) - \mathrm{I}\left(B; C\right)$$

$$= \mathrm{H}\left(A\right) - \mathrm{H}\left(B\right) - \mathrm{H}\left(A, C\right) + \mathrm{H}\left(B, C\right)$$

$$\leq \mathrm{H}\left(A \mid B\right) \quad \text{[because } B \leq A\text{].}$$

The equality holds if and only if $\mathrm{H}\left(A, C\right) = \mathrm{H}\left(B, C\right)$. Since $B \leq A$, it follows that $A + C = B + C$. □

Producing inequalities

Let B be the following 10×10 binary matrix

$$\begin{array}{cccccccccc} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \end{array}$$
$$\left(\begin{array}{cccccccccc}
0 & 1 & 1 & & & & & & & \\
1 & 0 & 1 & & O_{3\times3} & & & O_{3\times4} & & \\
1 & 1 & 0 & & & & & & & \\
& & & 0 & 1 & 1 & & & & \\
& O_{3\times3} & & 1 & 0 & 1 & & O_{3\times4} & & \\
& & & 1 & 1 & 0 & & & & \\
& & & & & & 0 & 1 & 1 & 1 \\
& & & & & & 1 & 0 & 1 & 1 \\
& O_{4\times3} & & & O_{4\times3} & & 1 & 1 & 0 & 1 \\
& & & & & & 1 & 1 & 1 & 0
\end{array}\right).$$

We calculate the rank of the matrix B over different fields to find:

$$\mathrm{rank}\left(B\right) = \begin{cases} 8, & \mathrm{char}\left(\mathbb{F}\right) = 2 \\ 9, & \mathrm{char}\left(\mathbb{F}\right) = 3 \\ 10, & \text{otherwise.} \end{cases}$$

For a column b_i of B, the set $\left\{j : b_i^j = 1\right\} \subseteq [10]$ is denoted by $\bar{b}_i$; if there is no confusion, by abuse of notation, we identify b_i and $\bar{b}_i$. The notation is the same for row and column vectors. We define:

$$\Delta_B\left(C\right) := \mathrm{H}\left(C \mid A_{[10]}\right) + \sum_{i=1}^{10} \mathrm{I}\left(A_{[n]-i}; C\right),$$

$$\Delta_B\left(A_{[10]}\right) := \mathrm{I}\left(A_1; A_2\right) + 2\mathrm{I}\left(A_{[3]}; A_{[4,10]}\right)$$
$$+\mathrm{I}\left(A_{[2]}; A_{[3,10]}\right) + 2\mathrm{I}\left(A_{[6]}; A_{[7,10]}\right) + \mathrm{I}\left(A_{[4]}; A_5\right)$$
$$+\mathrm{I}\left(A_{[5]}; A_{[6,10]}\right) + \sum_{i=8}^{10} \mathrm{I}\left(A_{[i-1]}; A_i\right),$$

$$\nabla_B\left(A_{[10]}\right) :=$$

$$\sum_{(j,k)\in S} \sum_{i=j}^{k} \left[\mathrm{I}\left(A_{[j-1]}; A_{[j,i-1]}\right) + \mathrm{I}\left(A_{[i]}; A_{[i+1,k]}\right)\right],$$

where S is the set of the three points $(1, 3)$, $(4, 6)$, $(7, 10)$.

The theorem of this paper shows three characteristic-dependent linear rank inequalities. The proof is guided by the matrix B; we choose this matrix because it is the smallest binary matrix, which we find, whose rank is different over at least three different finite fields. We hope that the arguments presented in the demonstration can be used to produce other inequalities taking matrices that have similar properties to this matrix.

Theorem 5. *Let A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, C be vector subspaces of a finite dimensional vector space V over a finite field $\mathbb{F}$. The following inequalities are characteristic-dependent linear rank inequalities:*

- If the characteristic of $\mathbb{F}$ is 2, then

$$\mathrm{H}\left(B_{[10]}\right) \leq 8\mathrm{I}\left(A_{[10]}; C\right) + \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_i, C : j \notin b_i\right) + 10\Delta_B\left(C\right)$$

$$+\Delta_B\left(A_{[10]}\right) + \nabla_B\left(A_{[10]}\right).$$

- If the characteristic of $\mathbb{F}$ is 2 or 3, then

$$\mathrm{H}\left(B_{[10]}\right) \leq 9\mathrm{I}\left(A_{[10]}; C\right) + \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_i, C : j \notin b_i\right) + 10\Delta_B\left(C\right)$$

$$+\Delta_B\left(A_{[10]}\right) + \nabla_B\left(A_{[10]}\right).$$

- If the characteristic of $\mathbb{F}$ is neither 2 nor 3, then

$$\mathrm{H}\,(C) \leq \frac{1}{10}\mathrm{I}\,\left(B_{[10]}\right) + \sum_{i=1}^{10} \mathrm{H}\,\left(B_i \mid A_j : j \in b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\,\left(C \mid A_i, B_i : j \notin b_i\right) + \Delta_B\,(C)$$

$$+ \Delta_B\,\left(A_{[10]}\right) + \nabla_B\,\left(A_{[10]}\right).$$

We remark that these inequalities do not in general hold over other fields whose characteristic is different to the described characteristic. A counterexample would be in $V = \mathrm{GF}\,(p)^{10}$. Take the vector subspaces: $A_i = \langle e_i \rangle$, the span of each vector of the canonical basis in V; $B_i = \langle b_i \rangle$, the span of each column of the matrix B; and $C = \langle (1 \cdots 1) \rangle$, the span of the vector with 1 in all entries. Then, if $p \neq 2$, the first inequality does not hold; if $p \neq 2, 3$, the second inequality does not hold; if $p = 2$ or 3, the third inequality does not hold.

The proof is given at the end of the section. Before, we introduce some propositions and lemmas that will help its development.

Let $V = A_1 \oplus \cdots \oplus A_n$, and take a vector subspace C of V such that $A_1 + \cdots + A_{i-1} + C + A_{i+1} + \cdots + A_n$ is a direct sum for all i. We say that $(A_1, \ldots, A_n, C)$ is *a tuple of complementary vector subspaces in V*.

Proposition 6. *A subspace C as above described holds* $\mathrm{H}\,(\pi_I\,(C)) = \mathrm{H}\,(C) \leq \mathrm{H}\,(A_1)$, *for all* $\emptyset \neq I \subseteq [n]$.

Proof. Let $x \in C$ such that $\pi_I\,(x) = O$. So, $\sum_{i \in I} x_i = O$. Hence, $x \in \bigoplus_{i \notin I} A_i$. By property of the tuple of complementary vector subspaces, $x = O$. In other words, $\pi_I\,(C)$ and C are isomorphic or have the same dimension. $\square$

As a consequence a non-zero element of C can be written uniquely as a sum of non-zero elements of A_1, …, A_n.

Consider any $n \times m$ binary matrix B', with columns denoted by b_i. Let π_{b_i} be the I_i-projection of

$$V = A_1 \oplus \cdots \oplus A_n,$$

where

$$I_i := \bar{b}_i = \left\{ j \in [n] : b_i^j = 1 \right\}.$$

Take

$$b_{I_i} := \sum_j b_{I_i}^j e_j \in V,$$

where $(e_j)_j$ is the canonical basis in V; and $b_{I_i}^j$ is 1 if $e_j \in A_k$ for some k such that $b_k = 1$, and 0 in otherwise. If $x = \sum_j x_j e_j$, then we have

$$\pi_{b_i}\,(x) = b_{I_i} \cdot x := \sum_j b_{I_i}^j x_j e_j.$$

We have the following proposition.

Proposition 7. *Let $V = A_1 \oplus \cdots \oplus A_n$, $A_i \neq O$, and let B' be a $n \times m$ binary matrix. For all i and I, we have $b_i = \sum_{j \in I} \alpha_j b_j$ if and only if $\pi_{b_i} = \sum_{j \in I} \alpha_j \pi_{b_j}$.*

Proof. We note $b_i = \sum_{j \in I} \alpha_j b_j$ if and only if $b_{I_i} = \sum_{j \in I} \alpha_j b_{I_j}$. Now, let $b_i = \sum_{j \in I} \alpha_j b_j$. For $x \in V$,

$$\sum_{j \in I} \alpha_j \pi_{b_j}\,(x) = \sum_{j \in I} \alpha_j \left(b_{I_j} \cdot x\right)$$

$$= \sum_{j \in I} \left(\alpha_j b_{I_j} \cdot x\right)$$

$$= \left(\sum_{j \in I} \alpha_j b_{I_j}\right) \cdot x$$

$$= b_{I_i} \cdot x$$

$$= \pi_{b_i}\,(x).$$

The other implication is obtained from

$$\pi_{b_i}\,(1 \cdots 1) = \sum_{j \in I} \alpha_j \pi_{b_j}\,(1 \cdots 1).$$

$\square$

Example 8. Take $B' = \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$. In $V = \mathrm{GF}\,(2)^5$, define $A_1 = \langle e_1, e_4 \rangle$, $A_2 = \langle e_2, e_5 \rangle$, $A_3 = \langle e_3 \rangle$. We have $b_{I_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $b_{I_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $b_{I_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. One can check that $b_1 = b_2 + b_3$ and $b_{I_1} = b_{I_2} + b_{I_3}$.

As a consequence of Proposition 6, if each A_i and C are one-dimensional vector spaces, then C is isomorphic to the span of $(1 \cdots 1)$, and each $\pi_{b_i}\,(C)$ is isomorphic to $\langle b_{I_i} \rangle$. From this and the last proposition, we have the following proposition that is a stronger version of this fact:

Corollary 9. *Let $(A_1, \ldots, A_n, C)$ be a tuple of complementary vector subspaces in V over $\mathbb{F}$ with $C \neq O$, and let B' be a $n \times m$ binary matrix with columns denoted by b_i. We have $\{b_i\}_{i \in I}$ is an independent set if and only if $\sum_{j \in I} \pi_{b_j}\,(C)$ is a direct sum.*

If we take $B' := B$, the 10×10 binary matrix defined above, in the last corollary, we obtain:

Corollary 10. *If* $(A_1, \ldots, A_{10}, C)$ *is a tuple of complementary vector subspaces of* V*, then*

$$H\left(\pi_{b_i}(C) : i \in [10]\right) = \begin{cases} 8H(C), & \text{char}(\mathbb{F}) = 2 \\ 9H(C), & \text{char}(\mathbb{F}) = 3 \\ 10H(C), & \text{otherwise.} \end{cases}$$

In the rest of the paper, we only work with the matrix B. The Corollary 10 is used immediately to find three inequalities which are true over vector spaces that hold some conditions. We note that the desired conditions are described by the dependency relationships of the columns of B.

Proposition 11. *Let* A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, C *vector subspaces of a finite dimensional vector space* V *over a finite field* $\mathbb{F}$ *such that* $(A_1, \ldots, A_{10}, C)$ *is a tuple of complementary vector subspaces. Consider the following conditions:*

(i) $B_i \leq \bigoplus_{j \in b_i} A_j$.

(ii) $B_i \leq \left(\bigoplus_{j \notin b_i} A_j\right) \oplus C$.

(iii) $C \leq \left(\bigoplus_{j \notin b_i} A_j\right) + B_i$.

We have the following implications:

1. If conditions (i) and (ii) hold, and the characteristic of $\mathbb{F}$ *is 2, then*

$$H\left(B_{[10]}\right) \leq 8H(C).$$

2. If conditions (i) and (ii) hold, and the characteristic of $\mathbb{F}$ *is 3, then*

$$H\left(B_{[10]}\right) \leq 9H(C).$$

3. If conditions (i) and (iii) hold, and the characteristic of $\mathbb{F}$ *is neither 2 nor 3, then*

$$10H(C) \leq H\left(B_{[10]}\right).$$

Proof. To proof 1 and 2, we can use

$$\pi_{b_i}(C) = \left(\bigoplus_{j \in b_i} A_j\right) \cap \left(\left(\bigoplus_{j \notin b_i} A_j\right) \oplus C\right).$$

Therefore, using conditions (i) and (ii), we have $B_i \leq \pi_{b_i}(C)$. Then we apply the last corollary. To proof 3, using conditions (i) and (iii), we can derive $\pi_{b_i}(C) \leq B_i$. Then we apply the last corollary. $\square$

Define $\bar{A}_1 := A_1$, and for $i > 1$, take a vector subspace of A_i which is complementary to $\sum_{j<i} A_j$ in $\sum_{j \leq i} A_j$, denote this subspace by $\bar{A}_i$. We have

$$V' := \bar{A}_1 \oplus \cdots \oplus \bar{A}_{10} \leq V.$$

Let $C^{(0)} = V' \cap C$. Recursively, for i, denote by $C^{(i)}$, a subspace of $C^{(i-1)}$ which is a complementary subspace to

$$\bar{A}_1 \oplus \cdots \oplus \bar{A}_{i-1} \oplus \bar{A}_{i+1} \oplus \cdots \oplus \bar{A}_{10}$$

in

$$\bar{A}_1 \oplus \cdots \oplus \bar{A}_{i-1} \oplus C^{(i-1)} \oplus \bar{A}_{i+1} \oplus \cdots \oplus \bar{A}_{10}.$$

Let $\bar{C}$ denote the subspace $C^{(10)}$. The tuple $(\bar{A}_1, \ldots, \bar{A}_{10}, \bar{C})$ is not unique but from now on we fix one of these tuples.

Lemma 12. $(\bar{A}_1, \ldots, \bar{A}_{10}, \bar{C})$ *is a tuple of complementary vector subspaces in* V' *that holds the following inequalities:*

$$H\left(C \mid \bar{C}\right) \leq \Delta_B(C),$$

$$\sum_{i=1}^{10} H\left(A_j : j \notin b_i \mid \bar{A}_j : j \notin b_i\right) \leq \Delta_B\left(A_{[10]}\right),$$

$$\sum_{i=1}^{10} H\left(A_j : j \in b_i \mid \bar{A}_j : j \in b_i\right) \leq \nabla_B\left(A_{[10]}\right).$$

Proof. The given tuple is a tuple of complementary vector subspaces in V' by definition. Furthermore, using Lemma 3, we have the following inequalities:

$$H\left(C \mid C^{(0)}\right) \leq H\left(C \mid A_{[10]}\right),$$

$$H\left(C^{(0)} \mid \bar{C}\right) \leq \sum_{i=1}^{10} I\left(A_{[n]-i}; C\right),$$

$$H\left(A_{[j,k]} \mid \bar{A}_{[j,k]}\right) = I\left(A_{[1,j-1]}; A_{[j,k]}\right),$$
$$\text{for all } j, k,$$

$$H\left(A_{[j,k]-i} \mid \bar{A}_{[j,k]-i}\right) \leq I\left(A_{[j-1]}; A_{[j,i-1]}\right)$$
$$+ I\left(A_{[i]}; A_{[i+1,k]}\right),$$
$$\text{for } (j,k) \in S, i \in [j,k],$$

$$H\left(A_{i\cup[4,10]} \mid \bar{A}_{i\cup[4,10]}\right) \leq I\left(A_{i-1}; A_2\right) + I\left(A_{[3]}; A_{[4,10]}\right)$$
$$\text{for } i = 1, 2,$$

$$H\left(A_{[4]\cup[7,10]} \mid \bar{A}_{[4]\cup[7,10]}\right) \leq I\left(A_{[6]}; A_{[7,10]}\right),$$

$$H\left(A_{5\cup[3]\cup[7,10]} \mid \bar{A}_{5\cup[3]\cup[7,10]}\right) \leq I\left(A_{[4]}; A_5\right)$$
$$+ I\left(A_{[6]}; A_{[7,10]}\right),$$

$$H\left(A_{[3]\cup[6,10]} \mid \bar{A}_{[3]\cup[6,10]}\right) \leq I\left(A_{[5]}; A_{[6,10]}\right),$$

$$H\left(A_{i\cup[6]} \mid \bar{A}_{i\cup[6]}\right) \leq I\left(A_{[i-1]}; A_i\right),$$
$$\text{for } i = 8, 9, 10.$$

One can use all these inequalities along with the definitions of $\Delta_B(C)$, $\Delta_B\left(A_{[10]}\right)$, $\nabla_B\left(A_{[10]}\right)$ (these definitions were previously given) to obtain the described inequalities. $\square$

Lemma 13. *For each i, let*

$$\bar{B}_i := B_i \cap \left(\bigoplus_{j \in b_i} \bar{A}_j \right) \cap \left(\left(\bigoplus_{j \notin b_i} \bar{A}_j \right) \oplus \bar{C} \right).$$

We have the subspaces $\bar{A}_1, \ldots, \bar{A}_{10}, \bar{B}_1, \ldots, \bar{B}_{10}, \bar{C}$ satisfy conditions (i), (ii) of Proposition 11, and the following inequality holds

$$\mathrm{H}\left(B_i : i \in [10] \mid \bar{B}_i : i \in [10]\right) \leq \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_i, C : j \notin b_i\right) + 10 \Delta_B\left(C\right)$$

$$+ \Delta_B\left(A_{[10]}\right) + \nabla_B\left(A_{[10]}\right).$$

Proof. The subspaces satisfy conditions (i) and (ii) of Proposition 11 by definition. Also, we have

$$\mathrm{H}\left(B_i \mid \bar{B}_i\right) \leq \mathrm{H}\left(B_i \mid \bar{A}_j : j \in b_i\right)$$

$$+ \mathrm{H}\left(B_i \mid \bar{A}_j, \bar{C} : j \notin b_i\right)$$

$$[\text{definition of } \bar{B}_i]$$

$$= \mathrm{H}\left(B_i \mid A_j : j \in b_i\right) + \mathrm{H}\left(B_i \mid A_j, C : j \notin b_i\right)$$

$$+ \mathrm{H}\left(B_i \cap \left[\bigoplus_{j \in b_i} A_j\right] \mid B_i \cap \left[\bigoplus_{j \in b_i} \bar{A}_j\right]\right)$$

$$+ \mathrm{H}\left(B_i \cap \left[C \oplus \bigoplus_{j \notin b_i} A_j\right] \mid B_i \cap \left[\bar{C} \oplus \bigoplus_{j \notin b_i} \bar{A}_j\right]\right)$$

$$[\text{because } \bar{A}_j \leq A_j, \bar{C} \leq C]$$

$$\leq \mathrm{H}\left(B_i \mid A_j : j \in b_i\right) + \mathrm{H}\left(A_j : j \in b_i \mid \bar{A}_j : j \in b_i\right)$$

$$+ \mathrm{H}\left(B_i \mid A_j, C : j \notin b_i\right)$$

$$+ \mathrm{H}\left(A_j, C : j \notin b_i \mid \bar{A}_j, \bar{C} : j \notin b_i\right)$$

$$[\text{from Lemma 4}]$$

$$\leq \mathrm{H}\left(B_i \mid A_j : j \in b_i\right) + \mathrm{H}\left(B_i \mid A_j, C : j \notin b_i\right)$$

$$+ \mathrm{H}\left(A_j : j \in b_i \mid \bar{A}_j : j \in b_i\right) + \mathrm{H}\left(A_j \mid \bar{A}_j : j \notin b_i\right)$$

$$+ \mathrm{H}\left(C \mid \bar{C}\right) \qquad [\text{from Lemma 3}].$$

Then, we apply the inequalities from Lemma 12. $\qquad \square$

Lemma 14. *For each i, let*

$$\hat{B}_i := B_i \cap \left(\bigoplus_{j \in b_i} \bar{A}_j \right),$$

$$\hat{C} := \bar{C} \bigcap_{i=1}^{10} \left(\left(\bigoplus_{j \notin b_i} \bar{A}_j \right) + \hat{B}_i \right).$$

We have the subspaces $\bar{A}_1, \ldots, \bar{A}_{10}, \hat{B}_1, \ldots, \hat{B}_{10}, \hat{C}$ satisfy conditions (i), (iii) of Proposition 11, and the following inequality holds

$$\mathrm{H}\left(C \mid \hat{C}\right) \leq \sum_{i=1}^{10} \mathrm{H}\left(C \mid A_j, B_i : j \notin b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right) + \Delta_B\left(C\right)$$

$$+ \Delta_B\left(A_{[10]}\right) + \nabla_B\left(A_{[10]}\right).$$

Proof. The subspaces satisfy conditions (i) and (iii) of Proposition 11 by definition. Also, we have

$$\sum_{i=1}^{10} \mathrm{H}\left(B_i \mid \hat{B}_i\right) \leq \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right) + \nabla_B\left(A_{[10]}\right).$$

Hence,

$$\mathrm{H}\left(C \mid \hat{C}\right) = \mathrm{H}\left(C \mid \bar{C}\right) + \mathrm{H}\left(\bar{C} \mid \hat{C}\right)$$

$$\leq \mathrm{H}\left(C \mid \bar{C}\right) + \mathrm{H}\left(C \mid \hat{C}\right)$$

$$[\text{because } \hat{C} \leq \bar{C} \leq C]$$

$$\leq \mathrm{H}\left(C \mid \bar{C}\right) + \sum_{i=1}^{10} \mathrm{H}\left(C \mid \bar{A}_j, \hat{B}_i : j \notin b_i\right)$$

$$[\text{definition of } \bar{B}_i]$$

$$\leq \mathrm{H}\left(C \mid \bar{C}\right) + \sum_{i=1}^{10} \mathrm{H}\left(C \mid A_i, B_i : j \notin b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(C \cap \left[B_i + \bigoplus_{j \notin b_i} A_j\right] \mid C \cap \left[\hat{B}_i + \bigoplus_{j \notin b_i} \bar{A}_j\right]\right)$$

$$[\text{because } \bar{A}_j \leq A_j, \hat{B}_i \leq B_i]$$

$$\leq \mathrm{H}\left(C \mid \bar{C}\right) + \sum_{i=1}^{10} \mathrm{H}\left(C \mid A_i, B_i : j \notin b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(A_j, B_i : j \notin b_i \mid \bar{A}_j, \hat{B}_i : j \notin b_i\right)$$

$$[\text{from Lemma 4}]$$

$$\leq \mathrm{H}\left(C \mid \bar{C}\right) + \sum_{i=1}^{10} \mathrm{H}\left(C \mid A_i, B_i : j \notin b_i\right)$$

$$+ \sum_{i=1}^{10} \mathrm{H}\left(A_j : j \notin b_i \mid \bar{A}_j : j \notin b_i\right) + \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid \hat{B}_i\right)$$

$$[\text{from Lemma 3}].$$

Then, we apply the inequalities from Lemma 12 and the inequality presented at the beginning of the proof. $\qquad \square$

Proof of Theorem 5

To prove the inequality 1, from Lemma 13, the vector subspaces $\bar{A}_1, \ldots, \bar{A}_{10}, \bar{B}_1, \ldots, \bar{B}_{10}, \bar{C}$ satisfy Proposition 11 with condition 1 if the characteristic of the field $\mathbb{F}$ is 2. We have

$$\mathrm{H}\left(\bar{B}_{[10]}\right) \leq 8\mathrm{H}\left(\bar{C}\right).$$

One can note $\mathrm{H}\left(\bar{C}\right) \leq \mathrm{I}\left(A_{[10]}; C\right)$, and the inequality given in Lemma 13 can be write as

$$\mathrm{H}\left(B_{[10]}\right) - \sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right)$$

$$-\sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_i, C : j \notin b_i\right) - 10\Delta_B\left(C\right)$$

$$-\Delta_B\left(A_{[10]}\right) - \nabla_B\left(A_{[10]}\right) \leq \mathrm{H}\left(\bar{B}_{[10]}\right).$$

Using all these inequalities we obtain the desired inequality.

The inequality 2 is obtained in a similar way; and from the inequality 1, it is easy to note that the inequality 2 also holds over fields whose characteristic is 2.

To prove the inequality 3, from Lemma 14, the vector subspaces $\bar{A}_1, \ldots, \bar{A}_{10}, \hat{B}_1, \ldots, \hat{B}_{10}, \hat{C}$ satisfy Proposition 11 with condition 3 if the characteristic of the field $\mathbb{F}$ is neither 2 nor 3, we have

$$10\mathrm{H}\left(\hat{C}\right) \leq \mathrm{H}\left(\hat{B}_{[10]}\right).$$

One can note $\mathrm{H}\left(\hat{B}_{[10]}\right) \leq \mathrm{H}\left(B_{[10]}\right)$, and the inequality given in Lemma 14 can be write as

$$\mathrm{H}\left(C\right) - \sum_{i=1}^{10} \mathrm{H}\left(C \mid A_j, B_i : j \notin b_i\right)$$

$$-\sum_{i=1}^{10} \mathrm{H}\left(B_i \mid A_j : j \in b_i\right) - \Delta_B\left(C\right)$$

$$-\Delta_B\left(A_{[10]}\right) - \nabla_B\left(A_{[10]}\right) \leq \mathrm{H}\left(\hat{C}\right).$$

Using all these inequalities we obtain the desired inequality.

$\square$

Conclusions

In this paper, we produce three characteristic-dependent linear rank inequalities in 21 variables using as guide a suitable binary matrix whose rank is different over at least three different fields (specifically, the rank depends on the characteristic of the field). The first inequality is true over fields whose characteristic is 2; the second inequality is true over fields whose characteristic is 2 or 3; and the third inequality is true over field whose characteristic is neither 2 nor 3. We hope that the technique presented here can be used to produce other inequalities, choosing other suitable matrices. In future work, the independence or dependence of these inequalities and their possible applications to Network Coding must be studied.

Contribution of the authors

The first author created the central idea of the manuscript and both authors contributed to its development and writing.

Acknowledgments

Conflict of interest

The authors declare that they have no conflict of interest.

References

Blasiak A., Kleinberg R., Lubetzky E. (2011). Lexicographic products and the power of non-Linear Network Coding. Foundations of Computer Science (FOCS) 2011 IEEE 52nd Annual Symposium on. 609-618.

Dougherty R., Freiling C., Zeger K. (2005). Insufficiency of linear coding in network information flow. IEEE Transactions on Information Theory. **51** (8): 2745-2759.

Dougherty R., Freiling C., Zeger K. (2013). Achievable rate regions for Network Coding. IEEE Transactions on Information Theory. **61** (5): 2488-2509.

Freiling E.F. (2014). Characteristic dependent linear rank inequalities and applications to Network Coding. Ph.D. thesis. San Diego, The United States: University of California.

Ingleton W. (1969). Representation of matroids. Combinatorial mathematics and its applications. Oxford. 149-167.

Kinser R. (2011). New inequalities for subspace arrangements. Journal Combinatorial Theory Serie A. **118** (1): 152-161.

Shen A., Hammer D., Romashchenko A.E., Vereshchagin N.K. (2000). Inequalities for Shannon entropy and Kolmogorov complexity. Journal of Computer and Systems Sciences. **60**: 442-464.