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Original article

Mathematics

Computation of matrices and submodular functions values associated to finite topological spaces

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Abstract

The submodular functions have shown their importance in the study and characterization of multiple properties of finite topological spaces, from numeric values provided by such functions (**Sarria, Roa & Varela**, 2014). However, the calculation of these values has been performed manually or even using Hasse diagrams, which is not practical. In this article, we present some algorithms that let us calculate some kind of polymatroid functions, specifically $f_{\mathcal{U}}$, $f_{\mathcal{D}}$, $\overline{f}$ and r_A, which define a topology, by using topogenous matrices.

Key words: Finite topological spaces, submodular functions, topogenous matrix, Stong matrix.

Cálculo de matrices y funciones submodulares asociadas a espacios topológicos finitos

Resumen

Las funciones submodulares han mostrado su importancia en el estudio y la caracterización múltiples propiedades de los espacios topológicos finitos, a partir de valores numéricos proporcionados por dichas funciones (**Sarria, Roa & Varela**, 2014). Sin embargo, el cálculo de éstos valores se ha realizado manualmente e incluso haciendo uso de diagramas de Hasse, lo que no es práctico. En este artículo, presentamos algunos algoritmos que nos permiten calcular cierta clase de funciones polimatroides, específicamente $f_{\mathcal{U}}$, $f_{\mathcal{D}}$, $\overline{f}$ y r_A, las cuales definen una topología, por medio del uso de matrices topogéneas.

Palabras clave: Espacios topológicos finitos, funciones submodulares, matriz topogénea, matriz de Stong.

Introduction

Alexandroff proved that finite topological spaces are in correspondence one-to-one with finite preorders (**Alexandroff**, 1937), showing that such spaces can be viewed from other mathematical structures. Likewise, Shiraki named as *topogenous matrices* the objects worked by Krishnamurthy, in an attempt to count the topologies that can be defined on a finite set (**Krishnamurthy**, 1966), a problem still unsolved. Moreover, such matrices provide all the information about the topology of a finite space (**Shiraki**, 1968), showing the relevance of these objects in the topological context.

Recently, connections between submodular functions and finite topological spaces have been developed (**Abril**, 2015), (**Sarria et al.**, 2014), allowing to interpret many topological concepts through numeric values provided by such associated functions, which is really important if we want to mechanize the verification of topological properties on subsets of an arbitrary finite space.

In view of the above, we regard some matrices, as topogenous and Stong matrices defined in sections 2 and 3, which are useful, for example, in the study of lattice of all topologies on a fixed set, and we link such matrices with particular submodular functions ($f_{\mathcal{U}}$, $f_{\mathcal{D}}$, $\overline{f}$ and r_A), important in the finite topological spaces context, by algorithms created to improve the computations shown in (**Abril**, 2015) and (**Sarria et al.**, 2014).

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From now on, we shall use exclusively the symbol X to denote a set of n elements $X = \{x_1, \ldots, x_n\}$, unless explicitly stated otherwise. We define $I_n = \{1, 2, \ldots, n\}$ and for a permutation σ of I_n, we use P_σ to denote the matrix whose entries satisfy

$$[P_\sigma]_{ij} = \delta(i, \sigma(j))$$

where δ is the Kronecker delta.

Topogenous matrix

Given a finite topological space $(X, \mathscr{T})$, denote by U_k the minimal open set containing x_k:

$$U_k = \bigcap_{x_k \in E \in \mathscr{T}} E$$

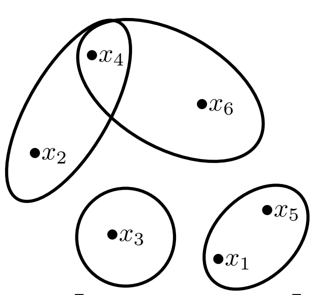
and consider the colection $\mathcal{U} = \{U_1, \ldots, U_n\}$, which is the minimal basis for the space in the sense that $\mathcal{U}$ is contained in any other basis for the topology $\mathscr{T}$.

Definition 0.1. Let $(X, \mathscr{T})$ be a finite topological space and $\mathcal{U} = \{U_1, \ldots, U_n\}$ its minimal basis. The *topogenous matrix* $T_X = [t_{ij}]$ associated to X is the square matrix of size $n \times n$ that satisfies:

$$t_{ij} = \begin{cases} 1 & , x_i \in U_j \\ 0 & , \textit{in other case} \end{cases}$$

Remark 0.2. (**Shiraki**, 1968) introduce the term *topogenous matrix* to denote the transpose matrix of that in above definition.

Example 0.3. In the next diagram, we represent the minimal basis for a topology on $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. The minimal open sets are $U_1 = U_5 = \{x_1, x_5\}, U_2 = \{x_2, x_4\}, U_3 = \{x_3\}, U_4 = \{x_4\}, U_6 = \{x_4, x_6\}$ and the associated topogenous matrix is as follows:



$$T_X = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 0.4. Consider the topological space $(X, \mathscr{T})$ where $X = \{a, b, c, d, e\}$ and

$$\mathscr{T} = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{d, e\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}, \{a, b, c, d\}, X\}$$

For the following orderings of the elements, we obtain the respective topogenous matrices as can be verified by calculating the minimal basis in each case:

$$(x_1, x_2, x_3, x_4, x_5) = (a, b, c, d, e)$$

$$T_{X_1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(x_1, x_2, x_3, x_4, x_5) = (b, d, e, a, c)$$

$$T_{X_2} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Topogenous matrices can be characterized by the following result.

Theorem 0.5. *(**Shiraki**, 1968) Let $T = [t_{ij}]$ be the topogenous matrix associated to $(X, \mathscr{T})$. Then T satisfies the following properties, for all $i, j, k \in I_n$:*

1. $t_{ij} \in \{0, 1\}$.
2. $t_{ii} = 1$.
3. $t_{ik} = t_{kj} = 1 \Longrightarrow t_{ij} = 1$.

Conversely, if a square matrix $T = [t_{ij}]$ of size $n \times n$ satisfies the above properties, T induces a topology on X.

Homeomorphism classes are also described by similarity via a permutation matrix between topogenous matrices.

Theorem 0.6. *(**Shiraki**, 1968) Let $(X, \mathscr{T})$ and $(Y, \mathscr{H})$ be finite topological spaces with associated topogenous matrices T_X and T_Y, respectively. Then $(X, \mathscr{T})$ and $(Y, \mathscr{H})$ are homeomorphic spaces if, and only if, T_X and T_Y are similar via a permutation matrix.*

Transiting Top(X)

Using topogenous matrices, we can find a way to *transit* through Top(X), the complete lattice of all topologies on a fixed set X. Given two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ in Top(X), the *supremum* of them, denoted by $\langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle$, is the topology whose open sets are unions of finite intersections of elements in the collection $\mathscr{T}_1 \cup \mathscr{T}_2$.

Proposition 0.7. *(**Cuevas**, 2016) Let $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ be finite topological spaces with minimal basis $\mathcal{U} = \{U_1, \ldots, U_n\}$ and $\mathcal{V} = \{V_1, \ldots, V_n\}$, respectively. Then, the minimal basis $\mathcal{W} = \{W_1, \ldots, W_n\}$ for the space $(X, \mathscr{T}^*)$, where $\mathscr{T}^* = \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle$, satisfies $W_k = U_k \cap V_k$ for all $k \in I_n$.*

The next theorem shows a way to *move ahead* in $\mathrm{Top}(X)$, that is, it allows to find the supremum of two topologies. The symbol $\wedge$ is regarded in the following sense: if E and F are $n \times n$ matrices, $E \wedge F$ is the square matrix whose entries satisfy $[E \wedge F]_{ij} = \min\{[E]_{ij}, [F]_{ij}\}$.

Theorem 0.8. *Let $X_1 = (X, \mathscr{T}_1)$, $X_2 = (X, \mathscr{T}_2)$ and $X^* = (X, \mathscr{T}^*)$ be finite topological spaces with topogenous matrices T_{X_1}, T_{X_2} and T_{X^*}, respectively. If $\mathscr{T}^* = \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle$ then*

$$T_{X^*} = T_{X_1} \wedge T_{X_2} \tag{1}$$

Conversely, if there exist finite spaces X_1, X_2 and X^* which satisfy (1) then $\mathscr{T}^* = \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle$.

Proof. Suppose that $\mathscr{T}^* = \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle$ and fix an index $k \in I_n$. By proposition 0.7, we know that $W_k = U_k \cap V_k$, thus $x_i \in W_k \iff x_i \in U_k$ and $x_i \in V_k$. Since the column k of a topogenous matrix represents the minimal open set which contains x_k, then for each $i \in I_n$ it is satisfied that $[T_{X^*}]_{ik} = \min\{[T_{X_1}]_{ik}, [T_{X_2}]_{ik}\}$.

The second part of the theorem holds by uniqueness of the minimal basis for a topology, since if (1) is satisfied, by the above argument we would have $T_{X^*} = T_{X_1} \wedge T_{X_2} = T_{(X, \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle)}$ and thus $\mathscr{T}^* = \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle$. ∎

A way to *go back* in $\mathrm{Top}(X)$ is finding subtopologies, which is possible using topogenous matrices as is shown in the next result.

Corollary 0.9. *Let $X_1 = (X, \mathscr{T}_1)$ and $X_2 = (X, \mathscr{T}_2)$ be finite topological spaces. Then $\mathscr{T}_1 \subseteq \mathscr{T}_2$ if, and only if, $[T_{X_2}]_{ij} \leqslant [T_{X_1}]_{ij}$ for all $i, j \in I_n$.*

Proof. The result follows from the next chain of equivalences:

$$\begin{aligned}\mathscr{T}_1 \subseteq \mathscr{T}_2 &\iff \mathscr{T}_2 = \langle \mathscr{T}_1 \cup \mathscr{T}_2 \rangle \\ &\iff T_{X_2} = T_{X_1} \wedge T_{X_2} \iff [T_{X_2}]_{ij} \leqslant [T_{X_1}]_{ij}\end{aligned}$$

∎

Triangularization of topogenous matrices

Let $(X, \mathscr{T})$ be a finite topological space with minimal basis $\mathcal{U} = \{U_1, \ldots, U_n\}$ and associated topogenous matrix $T_X = [t_{ij}]$. Define the binary relation $\leqslant$ on X as follows:

$$x_i \leqslant x_j \iff x_i \in U_j \iff U_i \subseteq U_j \iff t_{ij} = 1 \tag{2}$$

This relation is a preorder on the space, that is, it is reflexive and transitive. The next theorem shows under which condition such a relation is a partial order on X.

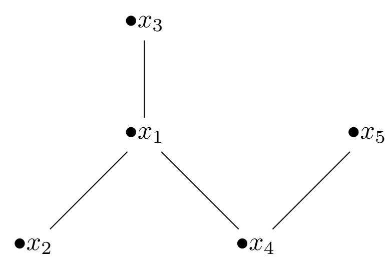
Theorem 0.10. *(**Alexandroff**, 1937) Topologies on a finite set X are in one-to-one correspondence with preorders on X. Moreover, a finite topological space $(X, \mathscr{T})$ is T_0 if, and only if, $(X, \leqslant)$ is a poset.*

Example 0.11. Consider the space $(X, \mathscr{T})$ given in the example 0.4 with the ordering

$$(x_1, x_2, x_3, x_4, x_5) = (a, b, c, d, e).$$

Such space satisfies the T_0 axiom. Hasse diagram of the poset $(X, \leqslant)$ is the next:

$$\begin{aligned} U_1 &= \{x_1, x_2, x_4\} \\ U_2 &= \{x_2\} \\ U_3 &= \{x_1, x_2, x_3, x_4\} \\ U_4 &= \{x_4\} \\ U_5 &= \{x_4, x_5\} \end{aligned}$$



In the example 0.4, it was possible to associate an upper triangular topogenous matrix to the considered space since such topological space is T_0, as shows the next Shiraki's theorem.

Theorem 0.12. *(**Shiraki**, 1968) A finite topological space $(X, \mathscr{T})$ is T_0 if, and only if, its associated topogenous matrix T_X is similar via a permutation matrix to an upper triangular topogenous matrix.*

A procedure to triangularize a topogenous matrix of a T_0 space X is described below. Given a topogenous matrix $T_X = [t_{ij}]$ define $M_k = \sum_{i=1}^{n} t_{ik} = |U_k|$, for each $k \in I_n$; if we organize them in ascending order

$$M_{k_1} \leqslant M_{k_2} \leqslant \cdots \leqslant M_{k_n} \tag{3}$$

and consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$, the topogenous matrix $P_\sigma^{\mathrm{T}} T_X P_\sigma$ is upper triangular: if $i > j$ and $t_{\sigma(i)\sigma(j)} = 1$, we would have $x_{\sigma(i)} < x_{\sigma(j)}$ and therefore $M_{k_i} < M_{k_j}$ which contradicts the ordering of M_k, hence $t_{\sigma(i)\sigma(j)} = 0$.

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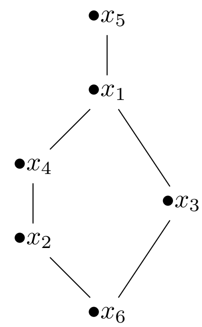
Example 0.13. Consider the topological space X, represented by the next Hasse diagram and topogenous matrix:



$$T_X = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

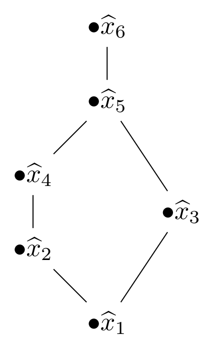
First, we find that: $M_1 = 5, M_2 = M_3 = 2, M_4 = 3, M_5 = 6$ and $M_6 = 1$. Therefore we obtain the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 3 & 4 & 1 & 5 \end{pmatrix} = (165)$$

whose associated matrix is given by

$$P_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we denote by $\widehat{x}_k = x_{\sigma(k)}$, the new ordering of the elements is represented in the topogenous matrix as follows:



$$P_\sigma^{\mathrm{T}} T_X P_\sigma = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 0.14. Observe that, in general, the permutation σ constructed using the relations in (3) is not unique. In example 0.13, we could have used $\sigma' = (165)(23)$ (and, in this case, no other!) to triangularize T_X obtaining the upper triangular matrix:

$$P_{\sigma'}^{\mathrm{T}} T_X P_{\sigma'} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Despite of this lack of uniqueness to choose such permutation σ, the resulting T_0 spaces are always homeomorphic (Theorem 0.6), so the topological properties are the same.

Remark 0.15. From now on, when $(X, \mathscr{T})$ is a T_0 space, we assume a fixed ordering in the elements of X such that its topogenous matrix T_X is upper triangular.

Stong matrix

Definition 0.16. Given X a T_0 space, we define the *Stong matrix* $S_X = [s_{ij}]$ as the square matrix of size $n \times n$ that satisfies

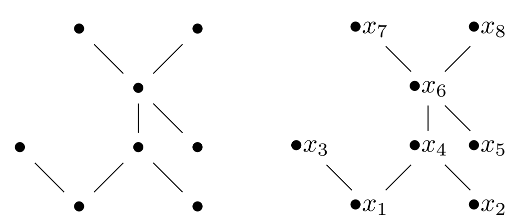
$$s_{ij} = \begin{cases} 1 & , x_i \leqslant x_j \textit{ and there is no } k \textit{ with } x_i < x_k < x_j \\ 0 & , \textit{in other case.} \end{cases}$$

A simple method to calculate the topogenous matrix T_X and the Stong matrix S_X of the space X, from the associated Hasse diagram, is described below. Number the vertices so that $x_i < x_j \Longrightarrow i < j$, that is, number them *from bottom to top* ensuring that the topogenous matrix is upper triangular. For each $i \neq j$:

- $t_{ij} = 1$ if, and only if, there exists a chain whose minimum is x_i and maximum is x_j.
- $s_{ij} = 1$ if, and only if, (x_i, x_j) is an edge of the diagram.

Remark 0.17. $t_{ij} = 0 \Rightarrow s_{ij} = 0$ and $s_{ij} = 1 \Rightarrow t_{ij} = 1$.

Example 0.18. Consider the Hasse diagram (left) numbering its vertices as described before (right):



For example, for x_5 we have $x_5 \leqslant x_5$, $x_5 \leqslant x_6$, $x_5 \leqslant x_7$ and $x_5 \leqslant x_8$ then the fifth row of T_X is [0 0 0 0 1 1 1 1]. Also, the edges with initial point x_1 are (x_1, x_3) and (x_1, x_4), then the first row of S_X is [1 0 1 1 0 0 0 0]. The associated matrices are the following:

$$T_X = \begin{bmatrix} 1&0&1&1&0&1&1&1\\ 0&1&0&1&0&1&1&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$S_X = \begin{bmatrix} 1&0&1&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

Relation between topogenous and Stong matrices

We can reconstruct the topogenous matrix from the Stong matrix and vice versa. In the first case, using theorem 0.5, we see that the topogenous matrix is the incidence matrix of the transitive closure of the binary relation represented by the Stong matrix. For the matrix given in example 0.18

$$S_X = [s_{ij}] = \begin{bmatrix} 1&0&1&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

we add ones in each upper subdiagonal. The first subdiagonal, consisting of the elements $s_{k,k+1}$, $1 \leqslant k \leqslant 7$, is not modified because if there is a k such that $s_{k,k+1} = 0$ and $t_{k,k+1} = 1$, there would exist x_j with $x_k < x_j < x_{k+1}$, which is not possible since the ordering we have chosen in remark 0.15 would imply $k < j < k+1$, a contradiction. For the second subdiagonal, whose elements are $s_{k,k+2}$, $1 \leqslant k \leqslant 6$, there are zeros in the entries s_{35} and s_{57}. Since $s_{56} = s_{67} = 1$ we have $t_{56} = t_{67} = 1 \Longrightarrow t_{57} = 1$ (theorem 0.5) and thus we add a one in the entry (5,7). In the case $s_{35} = 0$ there is no modification, because $s_{34} = s_{45} = 0$:

$$S_X^{(2)} = [s_{ij}^{(2)}] = \begin{bmatrix} 1&0&1&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&\mathbf{1}&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

For the third subdiagonal, with elements $s_{k,k+3}$, $1 \leqslant k \leqslant 5$, we have zeros in the entries $s_{25}^{(2)}$, $s_{36}^{(2)}$, $s_{47}^{(2)}$ and $s_{58}^{(2)}$. For example, since $s_{46}^{(2)} = s_{67}^{(2)} = 1$ then $t_{46} = t_{67} = 1 \Longrightarrow t_{47} = 1$, and hence we add a one in the entry $(4,7)$. Similarly for the entry $(5,8)$ since $s_{56}^{(2)} = s_{68}^{(2)} = 1$. Entry $(2,5)$ is not changed because there exists no k such that $s_{2k}^{(2)} = s_{k5}^{(2)} = 1$, and for the same reason we do not modify the entry $(3,6)$:

$$S_X^{(3)} = [s_{ij}^{(3)}] = \begin{bmatrix} 1&0&1&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&\mathbf{1}&0\\ 0&0&0&0&1&1&1&\mathbf{1}\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

Proceeding similarly for the following subdiagonals, we obtain the matrices $S_X^{(4)}$, $S_X^{(5)}$, $S_X^{(6)}$ and $S_X^{(7)}$:

$$S_X^{(4)} = \begin{bmatrix} 1&0&1&1&0&0&0&0\\ 0&1&0&1&0&\mathbf{1}&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&1&\mathbf{1}\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$S_X^{(5)} = \begin{bmatrix} 1&0&1&1&0&\mathbf{1}&0&0\\ 0&1&0&1&0&1&\mathbf{1}&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$S_X^{(6)} = \begin{bmatrix} 1&0&1&1&0&1&\mathbf{1}&0\\ 0&1&0&1&0&1&1&\mathbf{1}\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

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$$S_X^{(7)} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since we reached the last upper subdiagonal, we obtain the topogenous matrix $T_X = S_X^{(7)}$, which coincides with the matrix of example 0.18.

The above procedure can be applied to any finite T_0 space X by using the following algorithm.

Algorithm: Topogenous matrix from Stong matrix

Input: Stong matrix $S_X = [s_{ij}]_{n\times n}$ associated to the T_0 space X.

Output: Topogenous matrix $T_X = [t_{ij}]_{n\times n}$ associated to the T_0 space X.

1. **if** $n = 1, 2$ **do** $T = S$, **else**
2. **if** $j \leqslant i + 1$ or $s_{ij} = 1$ **do** $t_{ij} = s_{ij}$, **else**
3. **for** $j = 3 \cdots n$
4. **for** each zero element $s_{i,i+j-1}$ in the j-th upper subdiagonal
5. **if** there exists r such that $s_{ir} = 1 = s_{r,i+j-1}$ **do** $t_{i,i+j-1} = 1$

   else $t_{i,i+j-1} = 0$.
6. **end for**
7. **end for**

Reversing the above process, we obtain an algorithm to find the Stong matrix knowing the topogenous matrix of the space.

Algorithm: Stong matrix from topogenous matrix

Input: Topogenous matrix $T_X = [t_{ij}]_{n\times n}$ associated to the T_0 space X.

Output: Stong matrix $S_X = [s_{ij}]_{n\times n}$ associated to the T_0 space X.

1. **for** $i = 1 \cdots n$
2. **for** $j = 1 \cdots n$
3. **if** $j \leqslant i + 1$ or $t_{ij} = 0$ **do** $s_{ij} = t_{ij}$, **else**
4. **if** there exists k, between $i + 1$ and $j - 1$, such that $t_{ik} = 1 = t_{kj}$

   do $s_{ij} = 0$, **else** $s_{ij} = 1$
5. **end for**
6. **end for**

Calculating submodular functions values

The entropy function and $\overline{f}$

The entropy function r_A defined on an arbitrary FD-relation is studied in (**Sarria et al.**, 2014), in an attempt to apply the information theory in different structures. For each $\mathcal{N}$ FD-relation, there exists an special set A such that $r_A \in \lambda(\mathcal{N})$, which in our particular case of topological FD-relations, allows to find a polymatroid that characterizes the topological properties of the space, as is shown in (**Sarria et al.**, 2014). Here $\lambda(\mathcal{N})$ is the set of submodular and non-decreasing functions such that $\mathcal{N}_f = \mathcal{N}$. We intend in this subsection to propose an algorithm to find the values of such entropy function in any subset.

We know that $E = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq X$ is a closed set if, and only if, it coincides with its closure which means that every adherent point of E is in E. Using the minimal basis $\mathcal{U} = \{U_x\}_{x\in X}$ of the space X, we have the following characterization:

$$E \text{ is a closed set of } X \iff U_x \cap E = \emptyset \text{ for all } x \notin E \quad (4)$$

Bearing in mind that each column k of the topogenous matrix represents the minimal open set U_k, we shall find one by one the adherent points of E, not in E, finding the smallest closed set C which contains it, using the characterization given in (4), by the following procedure: we take $C^{(1)} := E = \{x_{i_1}, \ldots, x_{i_r}\}$; for each x_k such that $k \notin \{i_1, \ldots, i_r\}$, we consider the k-th column of the topogenous matrix and check the elements in rows $i_1, \ldots, i_r$ in such column; if they are all zero, we take $C = C^{(1)}$, otherwise, there would exist j_1 such that x_{j_1} is an adherent point of E hence we take $C^{(2)} = E \cup \{x_{j_1}\}$. Now, for each x_k such that $k \notin \{i_1, \ldots, i_r, j_1\}$, we consider the k-th column of the topogenous matrix and check the elements in rows $i_1, \ldots, i_r, j_1$ in such column; if they are all zero, we take $C = C^{(2)}$, otherwise, there would exist j_2 such that x_{j_2} is an adherent point of E, hence we take $C^{(3)} = E \cup \{x_{j_1}, x_{j_2}\}$.

Proceeding recursively, we continue adding points to obtain a set $C^{(m)} = E \cup \{x_{j_1}, x_{j_2}, \ldots, x_{j_{m-1}}\}$ such that $C^{(m)} = X$, in which case the set E is dense in X, or $C^{(m)}$ is such that for all $k \notin \{i_1, \ldots, i_r, j_1, \ldots, j_{m-1}\}$, the k-th column has zeros in entries $i_1, \ldots, i_r, j_1, \ldots, j_{m-1}$, in which case $C = C^{(m)}$.

Example 0.19. We shall find the closure of $\{x_2, x_4\}$ in the topological space X whose topogenous matrix is the following:

$$T_X = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Initially, we take $C^{(1)} = \{x_2, x_4\}$. Since the first column has zeros in rows 2 and 4, the element x_1 is not added to $C^{(1)}$; the third column has a one in the second row and so we make $C^{(2)} = \{x_2, x_3, x_4\}$. Every time we add a point, we must look again at those elements that we had discarded in the previous step, in this case x_1. The first column has zeros in rows 2, 3 and 4, then x_1 is not added to $C^{(2)}$; finally the fifth column has zeros in 2, 3 and 4, so x_5 is not added to $C^{(2)}$, and as we have exhausted all points of space, we conclude that the closure of $\{x_2, x_4\}$ is $C = C^{(2)} = \{x_2, x_3, x_4\}$.

The above comments help find the closure of an arbitrary subset using the following algorithm.

Algorithm: CLOSURE OF A SUBSET $I \subseteq X$

Input: Topogenous matrix $T_X = [t_{ij}]_{n\times n}$ associated to the space X, a subset $I \subseteq X$.

Output: Closure of I: $c(I) = C$.

1. Define $C = I$ and $r = 1$.
2. **while** $r = 1$
3. **for** $x_i \notin C$
4. **if** $\sum_{x_k \in C} t_{ik} \neq 0$ **do** $C \leftarrow C \cup \{x_i\}$ and go to step 3.
5. **if** $i = \max\{j : x_j \notin C\}$ **do** $r = 0$.
6. **end for**
7. **end while**

Remark 0.20. In (**Sarria et al.**, 2014) is proved that the map

$$\begin{aligned} \overline{f} : 2^X &\longrightarrow \mathbb{Z} \\ I &\longrightarrow \overline{f}(I) = |c(I)| \end{aligned}$$

where c is the closure operator associated to the finite topological space $(X, \mathscr{T})$, is a polymatroid which satisfies $\overline{f} \in \lambda(\mathcal{N}_{\mathscr{T}})$ hence, in particular, the above algorithm allows to determine the function values of $\overline{f}$ completely.

Now, let $n_K = |c(K)|$ be the cardinal number of the closure of K. In (**Sarria et al.**, 2014) it is shown that if $F_K = 2^n - 2^{n_K}$, the entropy function $r_A : 2^X \longrightarrow \mathbb{R}$ satisfies

$$r_A(I) = \ln|A| - \frac{2S_I}{|A|}\ln 2$$

where

$$S_I = \sum_{I \subseteq c(K)} F_K$$

$$M_I = \sum_{I \nsubseteq c(K)} F_K$$

$$|A| = 2(M_I + S_I)$$

We can rewrite S_I and M_I as follows:

$$\begin{aligned} S_I = \sum_{I \subseteq c(K)} F_K &= \sum_{I \subseteq c(K)} (2^n - 2^{n_K}) \\ &= 2^n \sum_{I \subseteq c(K)} \left[1 - 2^{n_K - n}\right] =: 2^n S_I^* \end{aligned}$$

$$\begin{aligned} M_I = \sum_{I \nsubseteq c(K)} F_K &= \sum_{I \nsubseteq c(K)} (2^n - 2^{n_K}) \\ &= 2^n \sum_{I \nsubseteq c(K)} \left[1 - 2^{n_K - n}\right] =: 2^n M_I^* \end{aligned}$$

so that $|A| = 2^{n+1}(M_I^* + S_I^*)$ and hence

$$\begin{aligned} r_A(I) &= \ln|A| - \frac{2S_I}{|A|}\ln 2 \\ &= \ln(2^{n+1}(M_I^* + S_I^*)) - \frac{2^{n+1}S_I^*}{2^{n+1}(M_I^* + S_I^*)}\ln 2 \\ &= (n+1)\ln 2 + \ln(M_I^* + S_I^*) - \frac{S_I^*}{(M_I^* + S_I^*)}\ln 2 \\ &= \ln(M_I^* + S_I^*) + \left[n + \frac{M_I^*}{M_I^* + S_I^*}\right]\ln 2 \end{aligned}$$

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Having the algorithm to find the closure of a subset K, and therefore to calculate the values n_K, S_I^* and M_I^*, we can evaluate the function r_A by the following algorithm.

Algorithm: ENTROPY FUNCTION r_A VALUE IN A SUBSET $I \subseteq X$

Input: Topogenous matrix $T_X = [t_{ij}]_{n\times n}$ associated to the space X, a subset $I \subseteq X$.

Output: Value $r_A(I)$.

1. Define $M_I^* = S_I^* = 0$.
2. **for** $K \subseteq X$
3. Calculate $n_K = |c(K)|$.
4. **if** $I \subseteq c(K)$ **do** $S_I^* \leftarrow S_I^* + 1 - 2^{n_K - n}$
   else $M_I^* \leftarrow M_I^* + 1 - 2^{n_K - n}$
5. **end for**
6. Calculate $r_A(I) = \ln(M_I^* + S_I^*) + \left[n + \frac{M_I^*}{M_I^*+S_I^*}\right]\ln 2$

Functions $f_{\mathcal{U}}$ and $f_{\mathcal{D}}$: matrices U_X and D_X

Let $(X, \mathscr{T})$ be a finite topological space, $\mathcal{U}$ its minimal basis and $\mathcal{D}$ the minimal basis for X^{op}, the opposite space of X, and consider the non-decreasing submodular function $f_\Delta : 2^X \longrightarrow \mathbb{Z}$ defined by

$$f_\Delta(I) := \sum_{J\in\Delta} q_J(I)$$

where $\Delta \subseteq 2^X$ and $q_J : 2^X \to \{0,1\}$ is the map which satisfies

$$q_J(I) := \begin{cases} 1\ , & I \nsubseteq J \\ 0\ , & I \subseteq J \end{cases}$$

for each $I, J \subseteq X$. In (**Abril**, 2015) is proved that if B is a subset of X then

$$f_{\mathcal{U}}(B) = |X| - |B^\triangledown|$$
$$f_{\mathcal{D}}(B) = |X| - |B_\triangle|$$

where $B^\triangledown$ and $B_\triangle$ denote the sets of upper and lower bounds of B in $(X, \leqslant)$, respectively.

These functions $f_{\mathcal{U}}$ and $f_{\mathcal{D}}$ are important to describe the topology of the space, since they characterize the order relation $\leqslant$ given in (2):

$$x_i \leqslant x_j \iff f_{\mathcal{D}}(x_i) = f_{\mathcal{D}}(x_i, x_j) \qquad (5)$$

$$x_i \leqslant x_j \iff f_{\mathcal{U}}(x_j) = f_{\mathcal{U}}(x_i, x_j) \qquad (6)$$

Definition 0.21. Given a finite topological space X, we define the matrix $U_X = [u_{ij}]$ associated to the function $f_{\mathcal{U}}$, as the matrix which satisfies $u_{ij} = f_{\mathcal{U}}(x_i, x_j)$. Analogously, we define the matrix $D_X = [d_{ij}]$ associated to the function $f_{\mathcal{D}}$ as the matrix satisfying $d_{ij} = f_{\mathcal{D}}(x_i, x_j)$.

Proposition 0.22. *If T_X is the topogenous matrix associated to the finite topological space X, then*

$$U_X = n\mathbf{1} - T_X T_X^{\mathrm{T}}$$

$$D_X = n\mathbf{1} - T_X^{\mathrm{T}} T_X$$

where $\mathbf{1} = [a_{ij}]$ is the square matrix of size $n \times n$ such that $a_{ij} = 1$ for all $i, j \in I_n$.

Proof. If $E = \{x_i, x_j\}^\triangledown$, we show that $|E| = \sum_{k=1}^n t_{ik}t_{jk}$: in fact, we have the equivalences:

$$\begin{aligned} x_k \in E &\iff x_i \leqslant x_k \ y \ x_j \leqslant x_k \\ &\iff t_{ik} = 1 \ y \ t_{jk} = 1 \\ &\iff t_{ik}t_{jk} = 1 \end{aligned}$$

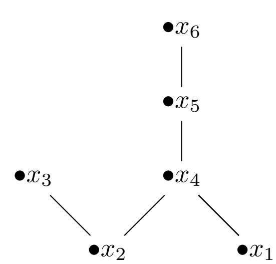
then each $x_k \in E$ provides a one in the sum $\sum_{k=1}^n t_{ik}t_{jk}$, thus having the required equality. Now, if $T_X T_X^{\mathrm{T}} = [c_{ij}]$ then $c_{ij} = \sum_{k=1}^n t_{ik}t_{jk} = |E|$ and hence

$$u_{ij} = f_{\mathcal{U}}(x_i, x_j) = |X| - |E| = n - c_{ij} = [n\mathbf{1} - T_X T_X^{\mathrm{T}}]_{ij}.$$

Let $A = \{x_i, x_j\}_\triangle$. By a similar argument above, we have $|A| = \sum_{k=1}^n t_{ki}t_{kj}$ from which we get

$$d_{ij} = f_{\mathcal{D}}(x_i, x_j) = |X| - |A| = [n\mathbf{1} - T_X^{\mathrm{T}} T_X]_{ij}. \quad \blacksquare$$

Example 0.23. For the next T_0 space



$$T_X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

matrices U_X and D_X are:

$$U_X = \begin{bmatrix} 2 & 3 & 6 & 3 & 4 & 5 \\ 3 & 1 & 5 & 3 & 4 & 5 \\ 6 & 5 & 5 & 6 & 6 & 6 \\ 3 & 3 & 6 & 3 & 4 & 5 \\ 4 & 4 & 6 & 4 & 4 & 5 \\ 5 & 5 & 6 & 5 & 5 & 5 \end{bmatrix}$$

$$D_X = \begin{bmatrix} 5 & 6 & 6 & 5 & 5 & 5 \\ 6 & 5 & 5 & 5 & 5 & 5 \\ 6 & 5 & 4 & 5 & 5 & 5 \\ 5 & 5 & 5 & 3 & 3 & 3 \\ 5 & 5 & 5 & 3 & 2 & 2 \\ 5 & 5 & 5 & 3 & 2 & 1 \end{bmatrix}$$

Proposition 0.24. *Let X be a finite topological space and $M = \{x_{i_1}, \dots, x_{i_m}\}$ a subset of X, then*

$$f_{\mathcal{U}}(M) = n - \sum_{r=1}^{n} \left(\prod_{k=1}^{m} t_{i_k r} \right)$$

$$f_{\mathcal{D}}(M) = n - \sum_{r=1}^{n} \left(\prod_{k=1}^{m} t_{r i_k} \right)$$

Proof.

$$\begin{aligned} x_r \in M^{\triangledown} &\iff x_{i_k} \leqslant x_r \text{ for all } k = 1, \dots, m \\ &\iff t_{i_k r} = 1 \text{ for all } k = 1, \dots, m \\ &\iff \prod_{k=1}^{m} t_{i_k r} = 1 \end{aligned}$$

$$\begin{aligned} x_r \in M_{\triangle} &\iff x_r \leqslant x_{i_k} \text{ for all } k = 1, \dots, m \\ &\iff t_{r i_k} = 1 \text{ for all } k = 1, \dots, m \\ &\iff \prod_{k=1}^{m} t_{r i_k} = 1 \end{aligned}$$

Therefore

$$|M^{\triangledown}| = \sum_{r=1}^{n} \left(\prod_{k=1}^{m} t_{i_k r} \right) \text{ and } |M_{\triangle}| = \sum_{r=1}^{n} \left(\prod_{k=1}^{m} t_{r i_k} \right).$$

■

Proposition 0.22 allows to obtain the matrices U_X and D_X from the topogenous matrix T_X. Consider now the reverse process of obtaining the topogenous matrix from U_X and from D_X. In our matrix language, (5) and (6) are equivalent to having

$$t_{ij} = 1 \iff u_{jj} = u_{ij} \iff d_{ii} = d_{ij}$$

Therefore, we can reconstruct the topogenous matrix by using the following algorithms:

Algorithm: TOPOGENOUS MATRIX FROM U_X

Input: Matrix $U_X = [u_{ij}]_{n \times n}$ associated to the function $f_{\mathcal{U}}$ of X.

Output: Topogenous matrix $T_X = [t_{ij}]_{n \times n}$ associated to the space X.

1. **for** $i = 1 \cdots n$
2. **for** $j = 1 \cdots n$
3. **if** $i > j$ **do** $t_{ij} = 0$, **else**
4. $t_{ij} = \delta(u_{ij} - u_{jj}, 0)$
5. **end for**
6. **end for**

Algorithm: TOPOGENOUS MATRIX FROM D_X

Input: Matrix $D_X = [d_{ij}]_{n \times n}$ associated to the function $f_{\mathcal{D}}$ of X.

Output: Topogenous matrix $T_X = [t_{ij}]_{n \times n}$ associated to the space X.

1. **for** $i = 1 \cdots n$
2. **for** $j = 1 \cdots n$
3. **if** $i > j$ **do** $t_{ij} = 0$, **else**
4. $t_{ij} = \delta(d_{ij} - d_{ii}, 0)$
5. **end for**
6. **end for**

Remark 0.25. From proposition 0.24 and above algorithms, we see that if $M = \{x_{i_1}, \dots, x_{i_m}\} \subseteq X$ we have:

$$f_{\mathcal{U}}(M) = n - \sum_{r=1}^{n} \left(\prod_{k=1}^{m} \delta(f_{\mathcal{U}}(x_r) - f_{\mathcal{U}}(x_{i_k}, x_r), 0) \right)$$

$$f_{\mathcal{D}}(M) = n - \sum_{r=1}^{n} \left(\prod_{k=1}^{m} \delta(f_{\mathcal{D}}(x_{i_k}) - f_{\mathcal{D}}(x_{i_k}, x_r), 0) \right)$$

Therefore, functions $f_{\mathcal{U}}$ and $f_{\mathcal{D}}$ can be calculated for any subset M of the space X, from the functions values in the subsets of cardinality one and two through the above explicit formulas.

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Conclusions

We have seen how the considered matrices make easier the topological study in finite spaces when we use submodular functions, achieving to eliminate the need to draw Hasse diagrams to find its values as was worked in (**Abril**, 2015) and (**Sarria et al.**, 2014). In future works, it can be studied complexity of the shown algorithms and try to find other topological concepts which could be characterized from this matrix point of view.

Conflict of interest. The authors declare that they have no conflict of interest.

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