ON DISTRIBUTIONAL REPRESENTATIONS OF MOMENT FUNCTIONALS: SIEVED POLLACZEC POLYNOMIALS

por

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Resumen

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Se establecen las representaciones distribucionales de las funcionales de momento de los polinomios cribados de Pollaczek de la primera y segunda clase. Estas representaciones subsisten para rangos más amplios de los parámetros que aquellos posibles obtenidos para representación por medidas positivas.

Abstract

Distributional representations of the moment functionals of the sieved Pollaczek polynomials of the first and second kinds are established. These representations hold for wider ranges of the parameters than those were the representation by positive measures is possible.

1. Introducción

A moment functional $\mathcal{L}$ ([11], Chap. I), i.e., a complex linear map of the space of complex polynomials into the field of complex numbers, is said to be regular, if it admits ;a system of monic *orthogonal polynomials*, a system $\{ P_n(x) | n > 0\}$ of complex polynomials satisfying a recurrence relation.

$$xP_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x),\ n > 0;$$

$$P_{-1}(x) = 0,\ P_o(x) = 1, \qquad (1.1)$$

with

$$C_{n+1} \neq 0, \qquad n \geq 0, \qquad (1.2)$$

such that

$$\mathcal{L}(P_o(x) = 1;\ L(P_n(x)) = 0,\ n \geq 1, \qquad (1.3)$$

and that

$$\mathcal{L}(P_n(x)\,P_m(x)) = \lambda n\delta mn,\ m,n \geq 0. \qquad (1.4)$$

with

$$\lambda o = 1;\ \lambda n = C_1 \ldots C_n,\ n \geq 1. \qquad (1.5)$$

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Observe that $\lambda_n \neq 0, \quad n \geq 0$. The system $\{P_n(x)\}$ is uniquely determined by $\mathcal{L}$ and is called the *monic orthogonal system of* $\mathcal{L}$. Conversely ([11],Chap.I), if a system of monic polynomials $\{P_n(x)\}$ is given by (1.1), if (1.2) holds and if $\mathcal{L}$ is defined through (1.3) and linear extension, then $\mathcal{L}$ is regular and $\{P_n(x)\}$ is its monic orthogonal system. The functional $\mathcal{L}$ is called the *moment functional of* $\{P_n(x)\}$.

The moment functional $\mathcal{L}$ is said to be *bounded* if there is a constant $M > 0$ such that for B_n, C_n in the recurrence relation (1.1) of its monic orthogonal system we have

$$|B_n| \leq \frac{M}{3}, \quad |C_{n+1}| \leq \frac{M^2}{9}, \; n \geq 0. \tag{1.6}$$

If $\mathcal{L}$ is regular and bounded (by M), the continued fraction

$$\frac{1 \;|}{|z - B_0} - \frac{C_1 \;|}{|z - B_1} - \frac{C_2 \;|}{|z - B_2} - \cdots \tag{1.7}$$

of its monic orthogonal system ([18], Chap.V) converges uniformly on $|z| \geq M'$,for all $M' > M$, to a limit $X(z)$, which is an analytic function on $|z| > M$. Then ([8], [12])

$$\mathcal{L}(P(x)) = \frac{1}{2\pi i}\int_C P(z)X(z)\,dz, \tag{1.8}$$

for any positively oriented contour of $|z| > M$ enclosing $z = 0$.

Representation (1.8) of $\mathcal{L}$ was established in [12] for special cases. A general proof based on the theory of continued fractions is in [8]. In the appendix at the end we include a proof based on functional analysis.

When $\mathcal{L}$ is *positive*, i.e., when B_n, C_n in (1.1) are real numbers and

$$C_{n+1} > 0, \quad n \geq 0, \tag{1.9}$$

$\mathcal{L}$ has the representation ([11], Chap. II)

$$\mathcal{L}(P(x)) = \int_{-\infty}^{\infty} P(x)\,d\mu(x), \tag{1.10}$$

where μ is a positive measure supported by the real line. If in addition (1.6) holds, μ is unique and $\operatorname{Supp}\mu \subseteq [-M, M]$. In these circumstances some powerful techniques have been deviced to determine μ explicitely. See [2] for many examples, and [5] for the special case of the Pollaczek polynomials. We mention that if (1.6) holds, (1.7) converges uniformly to $X(z)$ on any compact subset of $\mathbb{C} - [-M, M]$. If $\mathcal{L}$ is regular but not positive, representation (1.10) is impossible. However, (1.8) still holds if $\mathcal{L}$ is bounded.

Now assume $\mathcal{L}$ is regular and a polynomial

$$q(x) = a(x-\alpha_1)^{p_1}(x-\alpha_2)^{p_2}\ldots(x-\alpha_m)^{p_m} \tag{1.11}$$

with real roots $\alpha_1, \ldots, \alpha_m$ can be found such that the moment functional $\mathcal{U} = q(x)\mathcal{L}$ defined by

$$\mathcal{U}(P(x)) = \mathcal{L}(q(x)P(x)) \tag{1.12}$$

is positive, and let v be a positive measure representing $\mathcal{U}$ in the sense of (1.10). From the partial fraction decomposition

$$\frac{P(x)}{q(x)} = \sum_{i=1}^{m}\sum_{j=1}^{p_i}\frac{a_{ij}}{(x-\alpha_i)^j} + R_m(x), \tag{1.13}$$

$R_m(x)$ a polynomial, we deduce that

$$\mathcal{L}(P(x)) = \sum_{i=1}^{m}\sum_{j=1}^{p_i}\mathcal{L}\left(\frac{q(x)}{(x-\alpha_i)^j}\right)a_{ij} + \int_{-\infty}^{\infty} R_m(x)\,dv(x), \tag{1.14}$$

which is, since

$$a_{ij} = \frac{1}{(p_i - j)!}\frac{d^{p_i-j}}{dx^{p_i-j}}\left[\frac{P(x)(x-\alpha_i)^{p_i}}{q(x)}\right](\alpha_i), \tag{1.15}$$

a representation of $\mathcal{L}$ by distributions supported by the real line. Furthermore, if $\mathcal{L}$ is bounded and representation (1.8) holds, then

$$\mathcal{L}\left(\frac{q(x)}{(x-\alpha_i)^j}\right) = \frac{1}{2\pi i}\int_C \frac{q(z)X(z)}{(z-\alpha_i)^j}\,dz. \tag{1.16}$$

The above procedure is an alternative to that of Krall [13] and Morton and Krall [15] to establish distributional representations of regular moment functionals. It can be applied to some systems of polynomials which fall outside the scope of [13], [15]. This has been done in [8] to obtain distributional representations for the moment functional of the sieved ultraspherical polynomials and in [9] for that of the general Pollaczek polynomials, when the values of the parameters do not allow for representations by positive measures.

Our aim in this paper is to obtain explicit distributional representations for the moment functional of the sieved Pollaczek polynomials. Our approach differs from that followed in [9] for the Pollaczek polynomials in that contiguous function relations are favored over the theory of left multiplication of a regular functional by a polynomial. The approach in [9] would be too cumbersome if applied to the sieved Pollaczek polynomials. We also mention that the approach in [8] can not be followed in this case, as for the sieved Pollaczek polynomials there is no polynomial mapping involved.

For future reference we recall that the *Chebyshev polynomials of the first* and *second kinds* $\{T_n(x)\}$ and $\{U_n(x)\}$ are both defined (see [16], [17]) by the recurrence relation

$$2xy_n(x) = y_{n+1}(x) + y_{n-1}(x), \quad n \geq 1, \tag{1.17}$$

and the initial conditions are respectively $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1$, $U_1(x) = 2x$. For $x = \cos\theta$, $0 \leq \theta \leq \pi$, we have

$$\begin{aligned} T_n(x) &= \cos n\theta\ , \\ U_n(x) &= \frac{\sin(n+1)\theta}{\sin\theta}, \quad n \geq 0. \end{aligned} \tag{1.18}$$

We observe (with $T_{-1}(x) = 0 = U_{-1}(x)$) that

$$2T_n(x) = U_n(x) - U_{n-2}(x), \quad n \geq 1, \tag{1.19}$$

and also that

$$1 - T_n^2(x) = (1-x^2)U_{n-1}^2(x), \quad n \geq 0. \tag{1.20}$$

We also recall that the *hypergeometric series* is ([16], Chap. 4)

$$F\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, \quad |z| < 1, \tag{1.21}$$

where $(\alpha)_n$, given by $(\alpha)_0 = 1$, $(\alpha)_1 = \alpha$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n > 1$, is the *Pochhammer symbol*, so that c in (1.21) can not be zero nor a negative integer. The hypergeometric series defines an analytic function on $|z| < 1$. *The contiguous function relations* ([16], p.71)

$$(a-c+1)F\left(a,\ bc\,\middle|\, z\right) = aF\left(\begin{matrix} a+1,\ b \\ c \end{matrix}\,\middle|\, z\right) - (c-1)F\left(\begin{matrix} a,\ b \\ c-1 \end{matrix}\,\middle|\, z\right), \tag{1.22}$$

$$(a+b-c)F\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, z\right) = a(1-z)F\left(\begin{matrix} a+1, b \\ c \end{matrix}\,\middle|\, z\right) - (c-b)F\left(\begin{matrix} a,\ b-1 \\ c+1 \end{matrix}\,\middle|\, z\right), \tag{1.23}$$

and

$$\begin{aligned} &[1 - b + (c-a-1)z]F\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, z\right) \\ &\quad = (c-b)F\left(\begin{matrix} a, b-1 \\ c \end{matrix}\,\middle|\, z\right) - (c-1)(1-z)F\left(\begin{matrix} a, b \\ c-1 \end{matrix}\,\middle|\, z\right), \end{aligned} \tag{1.24}$$

will be needed in Section 2.

2. SIEVED POLLACZEK POLYNOMIALS

The sieved Pollaczek polynomials were introduced in [5] where they are derived from the q-Pollaczek polynomials by a procedure enterely anologous to that followed in [2] to define the sieved ultraspherical polynomials. A different approach to sieved polynomials is in [6], [7]. We here adopt this latter point of view.

Let $k \geq 2$ be an integer. The *k-sieved Pollaczek polynomials of the first kind* $\{P_n(k,\lambda,a,b;x)\}$ and *of the second kind* $\{Q_n(k,\lambda,a,b;x)\}$ are both defined by blocks of recurrence relations

$$\begin{aligned} xp_{nk+j}(x) &= p_{nk+j+1}(x) + B_n^{(j)} p_{nk+j}(x) \\ &\quad + C_n^{(j)} p_{nk+j-1}(x), \quad n \geq 0, \end{aligned} \tag{2.1}$$

$j = 0, 1, \ldots, k-1$, and initial conditions $p_{-1}(x) = 0$, $p_0(x) = 1$. For $\{P_n(k,\lambda,a,b;x)\}$, the coefficients $B_n^{(j)}$ and $C_n^{(j)}$, $n \geq 0$, are

$$\begin{aligned} B_n^{(0)} &= -\frac{b}{\lambda+a+n}; \\ B_n^{(j)} &= 0,\ j = 1, 2, \ldots, k-1, \\ C_n^{(0)} &= \frac{n}{4(\lambda+a+n)};\quad C_n^{(1)} = \frac{2\lambda+n}{4(\lambda+a+n)}, \\ C_n^{(j)} &= \frac{1}{4},\ j = 2, \ldots, k-1, \text{if } k > 2. \end{aligned} \tag{2.2}$$

The coefficients of $\{Q_n(k,\lambda,a,b;x)\}$ are, for $n \geq 0$,

$$\begin{aligned} B_n^{(j)} &= 0\ ,\ j = 0, 1, 2, \ldots, k-2, \\ B_n^{(k-1)} &= -\frac{b}{\lambda+a+n+1}, \\ C_n^{(0)} &= \frac{n}{4(\lambda+a+n)}; \end{aligned}$$

$$C_n^{(k-1)} = \frac{n+1+2\lambda}{4(\lambda+a+n+1)},$$
$$C_n^{(j)} = \frac{1}{4}\ ,\ j = 1,\dots,k-2, \text{if } k > 2. \tag{2.3}$$

It will be assumed throughout that

$$2\lambda \text{and } \lambda \pm a \text{are not integers} \leq 0. \tag{2.4}$$

This guaranties that (1.2) holds. To ensure (1.9), it has to be further assumed that λ, a, b are real numbers and that

$$\lambda > 0 \text{ and } \lambda + a > 0, \text{ or},$$
$$-\frac{1}{2} < \lambda < 0 \text{ and } 0 < \lambda + a + 1 < 1. \tag{2.5}$$

We will write $P_n^\lambda(x)$, $Q_n^\lambda(x)$ instead of $P_n(k,\lambda,a,b;x)$ and $Q_n(k,\lambda,a,b;x)$, respectively. In fact, throughout most arguments k, a, b will be kept fixed, and only λ should be emphasized.

Results in [5], [6] or [7] and some analytic continuation arguments (see [10] for details) give for the limits of the continued fractions of $\{P_n^\lambda(x)\}$ and $\{Q_n^\lambda(x)\}$ the expressions

$$X_\lambda(z) =$$
$$2(\lambda + a)U_{k-1}(z)\beta^k \frac{1}{-B_\lambda} F\left(\begin{matrix} A_\lambda + 1, 1 \\ -B_\lambda + 1 \end{matrix}\middle|\, \beta^{2k}\right) \tag{2.6}$$

and

$$\tilde{X}_\lambda(z) = 2\frac{U_{k-2}(z)}{U_{k-1}(z)} + \frac{2(1+2\lambda)\beta^k}{U_{k-1}(z)} \cdot \frac{1}{-B_\lambda + 1} F\left(\begin{matrix} A_\lambda,\ 1 \\ -B_\lambda + 2 \end{matrix}\middle|\, \beta^{2k}\right), \tag{2.7}$$

where

$$\alpha = z + (z^2-1)^{\frac{1}{2}},\ \beta = z - (z^2-1)^{\frac{1}{2}} \tag{2.8}$$

$$A_\lambda = -\lambda + \frac{az+b}{(z^2-1)^{\frac{1}{2}}},\ B_\lambda = -\lambda - \frac{az+b}{(z^2-1)^{\frac{1}{2}}}, \tag{2.9}$$

with $(z^2-1)^{\frac{1}{2}}$ denoting the branch of the square root of $z^2 - 1$ in $\mathbb{C}$ that behaves as z when $z \to \infty$. It can be shown that $(z^2-1)^{\frac{1}{2}}$ and thus $\alpha, \beta, A_\lambda, B_\lambda$ are analytic functions of z for $z \notin [-1,1]$. Furthermore, $\alpha + \beta = 2z$, $\alpha - \beta = 2(z^2-1)^{\frac{1}{2}}$, $\alpha\beta = 1$ and $|\beta| \leq 1 \leq |\alpha|$, with $|\beta| = |\alpha| = 1$ if and only if $z \in [-1,1]$. Thus, $X_\lambda(z)$ and $\tilde{X}_\lambda(z)$ are analytic functions of z on $\mathbb{C} - [-1,1]$, except for simple poles on the set

$$Z_\lambda = \{z \in \mathbb{C} - [-1,1] \mid B_\lambda(z) = 0, 1, 2, \dots\}. \tag{2.10}$$

Now, $B_\lambda(z) = n$ implies that

$$\left[(\lambda+n)^2 - a^2\right] z^2 - 2abz - b^2 - (n+\lambda)^2 = 0,$$
$$n \geq 0. \tag{2.11}$$

Hence, there are at most two values x_{2n} and x_{2n+1} of z at which $B_\lambda(z) = n$. With the determination of branch of the square root which is analytic in $\mathbb{C} - (-\infty, 0]$ and provided (2.4) holds, we can write

$$x_{2n} = \frac{ab - (n+\lambda)\sqrt{(n+\lambda)^2 + b^2 - a^2}}{(n+\lambda)^2 - a^2},$$
$$x_{2n+1} = \frac{ab + (n+\lambda)\sqrt{(n+\lambda)^2 + b^2 - a^2}}{(n+\lambda)^2 - a^2},$$
$$n \geq 0, \tag{2.12}$$

and observe that $x_{2n} \to -1$, $x_{2n+1} \to 1$ as $n \to +\infty$ and that x_0, x_1 are not poles of $\tilde{X}_\lambda(z)$.

We denote with $\mathcal{L}_\lambda$ the moment functional of $\{P_n^\lambda(x)\}$ and with $\tilde{\mathcal{L}}_\lambda$ that of $\{Q_n^\lambda(x)\}$. Provided $|z| > M_\lambda$, where

$$M_\lambda = 3\sup_n \left\{ \left| \frac{b}{n+a+\lambda} \right|, \sqrt{\left| \frac{n}{4(n+a+\lambda)} \right|},\ \sqrt{\left| \frac{n+2\lambda}{4(n+a+\lambda)} \right|} \right\}, \tag{2.13}$$

we have

$$\mathcal{L}_\lambda(P(x)) = \frac{1}{2\pi i}\int_C P(z)X_\lambda(z)\,dz \tag{2.14}$$

for C a positively oriented contour of $|z| > M_\lambda$ enclosing $z = 0$. Also

$$\tilde{\mathcal{L}}_\lambda(P(x)) = \frac{1}{2\pi i}\int_C P(z)\tilde{X}_\lambda(z)\,dz \tag{2.15}$$

for C in $|z| > \tilde{M}_\lambda$, where

$$\tilde{M}_\lambda = \sup_n \left\{ \left| \frac{b}{n+a+\lambda+1} \right|, \sqrt{\left| \frac{n+1}{4(n+a+\lambda+1)} \right|}, \sqrt{\left| \frac{n+2\lambda}{4(n+a+\lambda)} \right|} \right\}. \tag{2.16}$$

Observe that $\tilde{M}_\lambda \leq M_\lambda$. If λ, a, b are real numbers and (2.4), (2.5) hold, then (see [5]) $\mathcal{L}_\lambda$ can be represented by means of the positive measure

$$d\mu_\lambda(x) = \omega_\lambda(x)dx + \sum_{\zeta \in Z_\lambda} \operatorname{Res}(X_\lambda, \zeta)\delta(x-\zeta)dx \tag{2.17}$$

where

$$\omega_\lambda(x) = 2^{2\lambda-1}\frac{\lambda+a}{\pi\Gamma(2\lambda)}(1-x^2)^{\lambda-\frac{1}{2}}\left(U_{k-1}^2(x)\right)^\lambda\left|(1-\alpha^{2k})^{-(B_\lambda+\lambda)}\right|^2\left|\Gamma(-B_\lambda)\right|^2\chi(x), \tag{2.18}$$

χ being the characteristic function of $(-1,1)$, δ denoting the Dirac measure at $\zeta = 0$, and Z_λ being as in (2.10). Also, $\tilde{\mathcal{L}}_\lambda$ is represented (see [5]) by

$$d\tilde{\mu}_\lambda(x) = \tilde{\omega}_\lambda(x)dx + \sum_{\zeta \in \tilde{Z}_\lambda} \operatorname{Res}(\tilde{X}_\lambda, \zeta)\delta(x-\zeta)dx \tag{2.19}$$

where

$$\tilde{\omega}_\lambda(x) = \frac{2^{2\lambda+1}}{\pi\Gamma(2\lambda+1)}(1-x^2)^{\lambda+\frac{1}{2}}\left(U_{k-1}^2(x)\right)^\lambda \times\left|(1-\alpha^{2k})^{-(B_\lambda+\lambda)}\right|^2\left|\Gamma(-B_\lambda+1)\right|^2\chi(x), \tag{2.20}$$

and

$$\tilde{Z}_\lambda = \{z \in \mathbf{C} - [-1,1] \mid B_\lambda(z) = 1, 2, \ldots\}. \tag{2.21}$$

Explicit formulae for $\operatorname{Res}(X_\lambda, \zeta)$ and $\operatorname{Res}(\tilde{X}_\lambda, \zeta)$ can be found in [5]. We mention that Z_λ and $\tilde{Z}_\lambda$ can be empty, finite, or infinite countable with no limit point in $\mathbf{C} - [-1,1]$, according to the relative values of λ, a, b, and observe that $B_\lambda + \lambda$ is independent of λ. Let

$$Z = \{\lambda \in \mathbf{C} \mid 2\lambda \text{ or } \lambda \pm a \text{ is an integer} \leq 0\} \tag{2.22}$$

Lemma 2.1. *For λ not in Z,*

$$X_{\lambda+1}(z) = q_\lambda(z)X_\lambda(z) + r_\lambda(z), \quad |z| > \max\{M_\lambda, M_{\lambda+1}\}, \tag{2.23}$$

where

$$q_\lambda(x) = \frac{2(\lambda+a+1)}{\lambda(\lambda+a)(2\lambda+1)}(-A_\lambda)(-B_\lambda)(1-x^2)U_{k-1}^2(x) \tag{2.24}$$

and

$$r_\lambda(x) = -\frac{2(\lambda+a+1)}{\lambda(2\lambda+1)}[(ax+b)U_{k-1}(x) - \lambda T_k(x)]U_{k-1}(x) \tag{2.25}$$

are polynomials. Thus

$$\mathcal{L}_{\lambda+1} = q_\lambda(x)\mathcal{L}_\lambda \tag{2.26}$$

Proof. From (1.22), with $z = \beta^{2k}$, $a = A_\lambda = A$, $c = -B_{\lambda+2} = -B+2$ and $b = 1$, we obtain that

$$F\left(\begin{matrix}A,\ 1\\ -B+2\end{matrix}\,\middle|\,\beta^{2k}\right) = -\frac{1}{2\lambda+1}\left[AF\left(\begin{matrix}A+1,\ 1\\ -B+2\end{matrix}\,\middle|\,\beta^{2k}\right) - (-B+1)F\left(\begin{matrix}A,\ 1\\ -B+1\end{matrix}\,\middle|\,\beta^{2k}\right)\right]$$

and (1.23), (1.24) give, taking into account that

$$F\left(\begin{matrix}A\quad,\ 0\\ -B+2\end{matrix}\,\middle|\,\beta^{2k}\right) = F\left(\begin{matrix}A\quad,\ 0\\ -B+1\end{matrix}\,\middle|\,\beta^{2k}\right) \mathrel{\widehat{=}} 1,$$

that

$$F\left(\begin{matrix}A+1,\ 1\\ -B+2\end{matrix}\,\middle|\,\beta^{2k}\right) == \frac{-B+1}{2\lambda\beta^{2k}}\left[1-(1-\beta^{2k})F\left(\begin{matrix}A+1,\ 1\\ -B+1\end{matrix}\,\middle|\,\beta^{2k}\right)\right]$$

and

$$F\left(\begin{matrix}A\ ,1\\ -B+1\end{matrix}\,\middle|\,\beta^{2k}\right) == \frac{-1}{2\lambda}\left[A(1-\beta^{2k})F\left(\begin{matrix}A+1,\ 1\\ -B+1\end{matrix}\,\middle|\,\beta^{2k}\right) + B\right].$$

Hence

$$F\left(\begin{matrix} A,\ 1 \\ -B+2 \end{matrix}\,\middle|\,\beta^{2k}\right) = \frac{B-1}{2\lambda(2\lambda+1)}\left[\frac{A}{\beta^k}(\alpha^k - \beta^k) - 2\lambda - A(\alpha^k-\beta^k)^2 F\left(\begin{matrix} A+1,\ 1 \\ -B+1 \end{matrix}\,\middle|\,\beta^{2k}\right)\right], \quad (2.27)$$

and therefore

$$X_{\lambda+1} = -\frac{2(\lambda+a+1)}{2\lambda(2\lambda+1)}U_{k-1}(z)\left[A(\alpha^k-\beta^k) - 2\lambda\beta^k - A\beta^k(\alpha^k-\beta^k)^2 F\left(\begin{matrix} A+1,\ 1 \\ -B+1 \end{matrix}\,\middle|\,\beta^{2k}\right)\right].$$

Now we observe that from (1.18) and (2.8), $\alpha^k(z) = \alpha(T_k(z))$, $\beta^k(z) = \beta(T_k(z))$, so that $(\alpha^k-\beta^k)^2 = 4(T_k^2(z)-1) = 4(z^2-1)U_{k-1}^2(z)$

Here we use (1.20) and $\alpha^k + \beta^k = 2T_k(z)$. Hence, from (2.9), $A(\alpha^k - \beta^k) - 2\lambda\beta^k = 2[(az+b)U_{k-1}(z) - \lambda T_k(z)]$. Also,

$$q_\lambda(z) = -\frac{\lambda+a+1}{(\lambda+a)(2\lambda)(2\lambda+1)}(\alpha^k-\beta^k)^2(-A)(-B),$$

and taking into account (2.6), (2.24) follows at once. Since for a positively oriented contour C of $|z| > \max\{M_\lambda, M_{\lambda+1}\}$ enclosing $z=0$ we have

$$\begin{aligned}\mathcal{L}_{\lambda+1}(P(x)) &= \frac{1}{2\pi i}\int_C P(z)X_{\lambda+1}(z)\,dz \\ &= \frac{1}{2\pi i}\int_C P(z)q_\lambda(z)X_\lambda(z)\,dz \\ &\quad + \frac{1}{2\pi i}\int_C P(z)r_\lambda(z)\,dz \\ &= \frac{1}{2\pi i}\int_C P(z)q_\lambda(z)X_\lambda(z)\,dz \\ &= \mathcal{L}_\lambda(q_\lambda(x)P(x)),\end{aligned}$$

(2.26) holds. □

Now let

$$q_{m,\lambda}(x) = \frac{(\lambda+a+1)_m(-A_\lambda)_m(-B_\lambda)_m(1-x^2)^m U_{k-1}^{2m}(x)}{(\lambda)_m(\lambda+1/2)_m(\lambda+a)_m} \quad (2.28)$$

then $q_{0,\lambda}(x) = 1$, $q_{1,\lambda}(x) = q_\lambda(x)$. Also $q_{m,\lambda}(x) = q_\lambda(x)q_{\lambda+1}(x)\cdots q_{\lambda+m-1}(x)$ for $m > 1$, so that $q_{m,\lambda}(x)$ is a polynomial. Induction on Lemma 2.1 gives

Theorem 2.1. *For all $m \geq 0$ and λ not in Z,*

$$X_{\lambda+m}(z) = q_{m,\lambda}(z)X_\lambda(z) + r_{m,\lambda}(z), \quad |z| > \max\{M_\lambda, M_{\lambda+m}\}, \quad (2.29)$$

where $r_{m,\lambda}(x)$ is a polynomial. Furthermore

$$\mathcal{L}_{\lambda+m} = q_{m,\lambda}(x)\mathcal{L}_\lambda \quad (2.30)$$

From (2.7) and (2.27) we obtain, for $|z| > \tilde{M}_{\lambda-1}, \tilde{M}_\lambda$ and $\lambda \notin Z$, that

$$\begin{aligned}\tilde{X}_\lambda(z) &= 2\frac{U_{k-2}(z)}{U_{k-1}(z)} \\ &+ \frac{2\lambda+1}{U_{k-1}(z)}\cdot\frac{2}{-B+1}\left\{-\frac{(-B+1)}{(2\lambda)(2\lambda+1)}\left[A(\alpha^k-\beta^k) - 2\lambda\beta^k - A(\alpha^k-\beta^k)^2F\left(\begin{matrix} A+1,\ 1 \\ -B+1 \end{matrix}\,\middle|\,\beta^{2k}\right)\right]\right\} \\ &= 2\frac{U_{k-2}(z)}{U_{k-1}(z)} \\ &- \frac{1}{\lambda U_{k-1}(z)}\left\{2\left[(az+b)U_{k-1}(z) - \lambda T_k(z)\right] + 4(-A)(-B)(z^2-1)U_{k-1}^2(z)\frac{\beta^k}{-B}F\left(\begin{matrix} A+1,\ 1 \\ -B+1 \end{matrix}\,\middle|\,\beta^{2k}\right)\right\} \\ &= 2\frac{U_{k-2}(z)}{U_{k-1}(z)} + \frac{2T_k(z)}{U_{k-1}(z)} - \frac{2(az+b)}{\lambda} + \frac{4}{\lambda}(-A)(-B)(1-z^2)U_{k-1}(z)\frac{\beta^k}{-B}F\left(\begin{matrix} A+1,\ 1 \\ -B+1 \end{matrix}\,\middle|\,\beta^{2k}\right)\end{aligned}$$

Using (1.17) and (1.19) we get $T_k(x) = xU_{k-1}(x) - U_{k-2}(x)$, and simple calculations yield

$$\tilde{X}_\lambda(z) = \tilde{r}_{\lambda-1}(z) + \tilde{q}_{\lambda-1}\tilde{X}_{\lambda-1}$$

where

$$\tilde{q}_{\lambda-1}(x) = \frac{(-A)(-B)(1-x^2)U_{k-1}^2(x)}{\lambda(\lambda-1/2)}$$

and

$$\tilde{r}_{\lambda-1}(x) = 2\frac{(-A)(-B)(x^2-1)U_{k-2}(x)U_{k-1}(x)}{\lambda(\lambda-1/2)} - 2\frac{(a-\lambda)x+b}{\lambda}.$$

Thus, for λ not in Z,

$$\tilde{X}_{\lambda+1}(z) = \tilde{q}_\lambda(z)\tilde{X}_\lambda(z) + \tilde{r}_\lambda(z), \quad |z| > \max\{\tilde{M}_\lambda, \tilde{M}_{\lambda+1}\}, \tag{2.31}$$

where $\tilde{r}_\lambda(z)$ and $\tilde{q}_\lambda(z)$ are polynomials. Hence

$$\tilde{\mathcal{L}}_{\lambda+1} = \tilde{q}_\lambda(x)\tilde{\mathcal{L}}_\lambda. \tag{2.32}$$

Induction on (2.31) gives

Theorem 2.2. *For all* $m \geq 1$ *and* λ *not in* Z,

$$\tilde{X}_{\lambda+m}(z) = \tilde{r}_{m,\lambda}(z) + \tilde{q}_{m,\lambda}(z)\tilde{X}_\lambda(z), \quad |z| > \max\{\tilde{M}_\lambda, \tilde{M}_{\lambda+m}\}, \tag{2.33}$$

where

$$\tilde{q}_{m,\lambda}(z) = \frac{(-A_\lambda+1)_m(-B_\lambda+1)_m(1-x^2)^m U_{k-1}^{2m}(x)}{(\lambda+1)_m(\lambda+1/2)_m} \tag{2.34}$$

and $\tilde{r}_{m,\lambda}(x)$ *are polynomial. Furthermore*

$$\tilde{\mathcal{L}}_{\lambda+m} = \tilde{q}_{m,\lambda}(x)\tilde{\mathcal{L}}_\lambda. \tag{2.35}$$

Remark 2.1. Clearly $r_{m,\lambda}(x)$ and $\tilde{r}_{m,\lambda}(x)$ can be explicity calculated, but that information is not needed. From (2.6) and (2.7) we also get

$$\mathcal{L}_{\lambda+1} = \frac{\lambda+a+1}{2\lambda+1}\, U_{k-1}^2(x)\tilde{\mathcal{L}}_\lambda. \tag{2.36}$$

If $\mathcal{L}$ has a representation (1.8), we can define $q(x)\mathcal{L}$, where $q(x)$ is a rational function of x, in the obvious manner:

$$(q(x)\mathcal{L})(P(x)) = \frac{1}{2\pi i}\int_C q(z)P(z)X(z)\,dz, \tag{2.37}$$

provided all the poles of $q(z)$ and of $X(z)$ are within the contour C. From (2.36) it can be seen, however, that $\mathcal{U} = q(x)\mathcal{L}$ does not imply that $\mathcal{L} = (q(x))^{-1}\mathcal{U}$.

3. DISTRIBUTIONAL REPRESENTATIONS OF $\mathcal{L}_\lambda$ AND $\tilde{\mathcal{L}}_\lambda$

We assume λ ,a ,b to be real numbers and $m \geq 0$ to be such that

$$\lambda + m > 0 \quad \text{and} \quad \lambda + a + m > 0,$$

or,

$$-\frac{1}{2} < \lambda + m < 0 \quad \text{and} \quad 0 < \lambda + a + m < 1. \tag{3.1}$$

If (2.5) holds, m can be taken to be 0.

Let $-1 < \zeta_1 < \zeta_2 < \cdots < \zeta_{k-1} < 1$ be the roots of $U_{k-1}(x)$. Each is a root of multiplicity $2m$ of both $q_{m,\lambda}(x)$ and $\tilde{q}_{m,\lambda}(x)$. We assume $\alpha_1, \cdots, \alpha_n$ to be the other distinct roots of $q_{m,\lambda}(x)$, each of multiplicity m_j , $j = 1, 2, \cdots, n$. As for $\tilde{q}_{m,\lambda}(x)$, we denote with $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n$ its other distinct roots and with m_j the multiplicity of $\tilde{\alpha}_j$ (n and m_j do not need to be the same for $q_{m,\lambda}(x)$ and $\tilde{q}_{m,\lambda}(x)$). However, $m_1 + m_2 + ... + m_n = 2m$ in both cases. Since $A(\zeta_j)$, $B(\zeta_j)$ are either $-\lambda$ or a complex number, it follows that α_j is either x_{2i} or x_{2i+1} (as given by (2.12)) for some $i = 0, 1, \cdots, m-1$ (also $\tilde{\alpha}_j$ is x_{2i} or x_{2i+1} for some $i = 1, 2, \cdots, m$). From (1.14), (1.15), (1.16), (2.14) and (2.30) we obtain

Theorem 3.1. *If* λ, a, b *are real numbers, if (2.4) and (3.1) hold, and if*

$$(\lambda + j)^2 + b^2 \geq a^2,\ j = 0,\ 1,\ 2,\ \ldots,\ m-1, \tag{3.2}$$

then the moment functional $\mathcal{L}_\lambda$ *of* $\{P_n^\lambda(x)\}$ *has the distributional representation*

$$\mathcal{L}_\lambda = T_1 + T_2 + T_3, \tag{3.3}$$

where, for any test function φ *on the real line, we have*

$$T_1(\varphi) = \sum_{j=1}^{n}\sum_{h=1}^{m_j} A_{jh}\frac{d^{m_j-h}}{dx^{m_j-h}}\left[\frac{(x-\alpha_j)^{m_j}\varphi(x)}{q_{m,\lambda}(x)}\right](\alpha_j) \tag{3.4}$$

with

$$A_{jh} = \frac{1}{2\pi i(m_j-h)!}\int_C \frac{q_{m,\lambda}(z)X_\lambda(z)}{(z-\alpha_j)^h}\,dz, \tag{3.5}$$

and C *any positively oriented contour of* $|z| > \max\{M_\lambda,\ M_{\lambda+m}\}$ *enclosing* $z = 0$;

$$T_2(\varphi) = \sum_{j=1}^{k-1}\sum_{h=1}^{2m} A'_{jh}\frac{d^{2m-h}}{dx^{2m-h}}\left[\frac{(x-\zeta_j)^{2m}\varphi(x)}{q_{m,\lambda}(x)}\right](\zeta_j) \tag{3.6}$$

with

$$A'_{jh} = \frac{1}{2\pi i(2m-h)!}\int_C \frac{q_{m,\lambda}(z)X_\lambda(z)}{(z-\zeta_j)^h}\,dz, \tag{3.7}$$

$j = 1,\dots,k-1,\ h = 1,\dots,2m$; *and*

$$T_3(\varphi) = \int_{-\infty}^{+\infty} \varphi_m(x)\,d\mu_{\lambda+m}(x), \tag{3.8}$$

where

$$\begin{aligned}\varphi_m(x) = \frac{\varphi(x)}{q_{m,\lambda}(x)} &- \sum_{j=1}^{n}\sum_{h=0}^{m_j-1} \frac{1}{h!}\frac{d^h}{dx^h}\left[\frac{(x-\alpha_j)^{m_j}\varphi(x)}{q_{m,\lambda}(x)}\right](\alpha_j)\frac{1}{(x-\alpha_j)^{m_j-h}} \\ &- \sum_{j=1}^{k-1}\sum_{h=0}^{2m-1} \frac{1}{h!}\frac{d^h}{dx^h}\left[\frac{(x-\zeta_j)^{2m}\varphi(x)}{q_{m,\lambda}(x)}\right](\zeta_j)\frac{1}{(x-\zeta_j)^{2m-h}}.\end{aligned} \tag{3.9}$$

Furthermore,

$$\begin{aligned} &Supp\,T_1 = \{\alpha_j \mid j = 1,\dots,n\}, \\ &Supp\,T_2 = \{\zeta_1,\dots,\zeta_{k-1}\} \\ &Supp\,T_3 = Supp\,\mu_{\lambda+m} \end{aligned} \tag{3.10}$$

and thus T_1, T_2, T_3 are compactly supported on the real line and can act on polynomials.

Remark 3.1. That T_3 is a distribution follows from

$$|\varphi_m(x)| \le C\sum_{i=0}^{2m}]\sup_{t\in\mathbb{R}} \left|\varphi^{(i)}(t)\right|, \quad x \in [-M_{\lambda+m}, M_{\lambda+m}], \tag{3.11}$$

where $C > 0$ is a constant (independent of φ), which is a consequence of the Taylor Remainder Theorem.

Remark 3.2. If $m = 0$ in (3.1), i.e., if (2.5) is satisfied, then $T_1 = T_2 = 0$ and $T_3 = \mu_\lambda$. If (2.5) is not satisfied and $m > 0$, T_1 and T_2 measure the contribution to the orthogonality of the points α_j, $j = 1,\dots,n$, and of the points ζ_j, $j = 1,2,\dots,k-1$, where ω_λ becomes infinite.

Similarly, from (1.14), (1.15), (1.16), (2.32) and (2.35), we get

Theorem 3.2. *If λ, a, b are real numbers, if (2.4) and (3.1) hold, and if*

$$(\lambda+j)^2 + b^2 \ge a^2,\ j = 1,2,\dots,m, \tag{3.12}$$

then the moment functional $\tilde{\mathcal{L}}_\lambda$ of $\{Q_n^\lambda(x)\}$ has the distributional representation

$$\tilde{\mathcal{L}}_\lambda = \tilde T_1 + \tilde T_2 + \tilde T_3, \tag{3.13}$$

where, for any test function φ on the real line we have

$$\tilde T_1(\varphi) = \sum_{j=1}^{n}\sum_{h=1}^{m_j} \tilde A_{jh}\frac{d^{m_j-h}}{dx^{m_j-h}}\left[\frac{(x-\tilde\alpha_j)^{m_j}\varphi(x)}{\tilde q_{m,\lambda}(x)}\right](\tilde\alpha_j) \tag{3.14}$$

with

$$\tilde A_{jh} = \frac{1}{2\pi i(m_j-h)!}\int_C \frac{\tilde q_{m,\lambda}(z)\tilde X_\lambda(z)}{(z-\tilde\alpha_j)^h}\,dz, \quad j = 1,2,\dots,n\ ,\ h = 1,2,\dots,m_j \tag{3.15}$$

and C any positively oriented contour of $|z| > \max\{\tilde M_\lambda, \tilde M_{\lambda+m}\}$ enclosing $z = 0$;

$$\tilde T_2(\varphi) = \sum_{j=1}^{k-1}\sum_{h=1}^{2m} \tilde A'_{jh}\frac{d^{2m-h}}{dx^{2m-h}}\left[\frac{(x-\zeta_j)^{2m}\varphi(x)}{\tilde q_{m,\lambda}(x)}\right](\zeta_j) \tag{3.16}$$

with

$$\tilde A'_{jh} = \frac{1}{2\pi i(2m-h)!}\int_C \frac{\tilde q_{m,\lambda}(z)\tilde X_\lambda(z)}{(z-\zeta_j)^h}\,dz, \tag{3.17}$$

$j = 1,\dots,k-1,\ h = 1,2,\dots,2m$; *and*

$$\tilde T_3(\varphi) = \int_{-\infty}^{+\infty} \tilde\varphi_m(x)\,d\tilde\mu_{\lambda+m}(x), \tag{3.18}$$

where

$$\begin{aligned}\tilde\varphi_m(x) = \frac{\varphi(x)}{\tilde q_{\lambda,m}} &- \sum_{j=1}^{n}\sum_{h=0}^{m_j-1} \frac{1}{h!}\frac{d^h}{dx^h}\left[\frac{(x-\tilde\alpha_j)^{m_j}\varphi(x)}{\tilde q_{m,\lambda}(x)}\right](\tilde\alpha_j)\frac{1}{(x-\tilde\alpha_j)^{m_j-h}} \\ &- \sum_{j=1}^{k-1}\sum_{h=0}^{2m-1} \frac{1}{h!}\frac{d^h}{dx^h}\left[\frac{\varphi(x)(x-\zeta_j)^{2m}}{\tilde q_{m,\lambda}(x)}\right](\zeta_j)\frac{1}{(x-\zeta_j)^{2m-h}}.\end{aligned} \tag{3.19}$$

Moreover,

$$\begin{aligned} Supp\,\tilde{T}_1 &= \{\tilde{\alpha}_j \mid j = 1,2,\ldots,n\}, \\ Supp\,\tilde{T}_2 &= \{\zeta_1,\ldots,\zeta_{k-1}\} \\ Supp\,\tilde{T}_3 &= Supp\,\tilde{\mu}_{\lambda+m} \end{aligned} \tag{3.20}$$

and thus $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ have compact support on the real line and can act on polynomials.

Remark 3.3. Again $\tilde{T}_1 = \tilde{T}_2 = 0$ and $\tilde{T}_3 = \tilde{\mu}_\lambda$ if m can be taken to be 0.

Remark 3.4. If $a \neq \pm b$ and $(j+n)^2 + b^2 > a^2$ for $j = 0, 1, \ldots, m-1$, it can be shown (see [9]) that α_j has multiplicity 1 (so that $m_j = 1$ and $n = 2m$). If such is the case, T_1 in (3.3) is a measure. Also, if $a \neq \pm b$ and $(j+\lambda)^2 + b^2 > a^2$ for $j = 1, 2, \ldots, m$, $\tilde{T}_1$ in (3.13) is a measure.

Remark 3.5. If $a = b = 0$, $\{P_n^\lambda(x)\}$ and $\{Q_n^\lambda(x)\}$ are respectively the systems of sieved ultraspherical polynomials of the first and second kinds (see [1], [8]). Their distributional representations have been studied in [8]. We observe that in such case

$$\begin{aligned} \tilde{q}_{m,\lambda}(x) &= q_{m,\lambda}(x) \\ &= \frac{(\lambda+1)_m}{(\lambda+\frac{1}{2})_m}(1-x^2)^m U_{k-1}^{2m}(x), \end{aligned} \tag{3.21}$$

and its only roots are -1, 1 each of multiplicity m, and $\zeta_1, \ldots, \zeta_{k-1}$, each of multiplicity $2m$. It is easily seen that relations (3.3) and (3.13) respectively reduce to those in Theorems 5.1 and 5.2 of [8].

Remark 3.6. Extrapolating to $k = 1$ the distributional representation (3.3) of $\mathcal{L}_\lambda$ we obtain (2.26) of [9].

Remark 3.7. Now we observe that in spite of the apparent freedom of choice of m in Theorems 3.1 and 3.2, the distributional representation of $\mathcal{L}$ is unique, as far as only distributions with compact support are taken into account. This follows from general results (mainly due to H. G. Tillmann) on the theory of representations of distributions on the real line by analytic functions on $\mathbf{C} - \mathbf{R}$ (see [4], Chap 5). In fact, if T is a distribution with compact support K on $\mathbf{R}$, the *Cauchy-Stieljes transform of* T,

$$\hat{T}(z) = T_\zeta\left(\frac{1}{z-\zeta}\right), \tag{3.22}$$

is an analytic function off K, and if $K \subseteq (-M, M)$ and $|z| > 2M$, from the uniform convergence of $\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}}$ on $(-M, M)$ it follows that

$$\hat{T}(z) = \sum_{n=0}^{\infty} \frac{T(\zeta^n)}{z^{n+1}} \ . \tag{3.23}$$

Hence, if T represents $\mathcal{L}$, $T(\zeta^n) = \mathcal{L}(\zeta^n) = c_n$ is the n^{th}-moment of $\mathcal{L}$, and

$$\hat{T}(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}} = X(z)\ , \quad |z| > 2M, \tag{3.24}$$

where $X(z)$ is the limit of continued fraction of the monic orthogonal system of $\mathcal{L}$ (as in (1.34). For a proof of (3.24), see [18], Chap. XI or the appendix at end). Hence, $\hat{T}(z)$ is an analytic continuation of $X(z)$ from $|z| > 2M$ to $\mathbf{C} - K$. This implies, in view of the Stieljes inversion formula ([4], Chap. 5), that

$$< T, \varphi >= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \{X(x+i\epsilon) - X(x-i\epsilon)\}\varphi(x)dx \tag{3.25}$$

for any test function φ, which ensures the uniqueness of T.

Remark 3.8. Under the assumptions of each of Theorems 3.1, or 3.2, a measure on the line can be found which represents $\mathcal{L}$ (Boas [3]. See also [11], Chap. II). Since the distributions representing $\mathcal{L}$ in (3.3) or (3.13) are not measures when the positivity conditions fail, Boas' measures can not be supported by a compact set under such circumstances (as follows from the arguments in Remark 3.7).

4. APPENDIX

We include in this appendix a functional analytic proof of (1.8). To this purpose, let

$$J = \begin{pmatrix} B_0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ C_1 & B_1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & C_2 & B_2 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \end{pmatrix} \tag{4.1}$$

be the nfinite tri-diagonal matrix of the coefficients B_n, C_n in (1.1), and for each $n \geq 1$, let J_n be the submatrix of the first n rows and columns of J, and $\tilde{J}_n$, the infinite matrix

$$\begin{pmatrix} J_n & 0 \\ 0 & 0 \end{pmatrix} \tag{4.2}$$

Let I_n be the $n\times n$ identity matrix. From (1.1) it follows at once that

$$P_n(x)=Det(xI_n-J_n),\ n\geq 1. \qquad (4.3)$$

Also let J_n^* be the matrix obtained from J_n by deleting the first row and column, and define $\{P_n^*(x)\}$ by

$$\begin{aligned}&P_0^*(x)=0,\ P_1^*(x)=1;\\ &P_n^*(x)=Det(xI_{n-1}-J_n^*),\ n\geq 2.\end{aligned} \qquad (4.4)$$

It is readily verified that $\{P_n^*(x)\}$ satisfies (1.1) for $n\geq 1$.

Now let l_2 be the Hilbert space of square summable sequences $(x_0,x_1,\ldots,)$, with inner product

$$((x_n);(y_n))=\sum_{n=0}^{\infty}x_n\overline{y}_n$$

and norm $\sqrt{((x_n);(y_n))}$. With $\{e_n\mid n\geq 0\}$ we denote its cannonical orthonormal basis ($e_n=(\delta_{n0},\delta_{n1},\ldots,)$). If (1.6) holds, a bounded operator L on l_2 is defined by

$$Le_n=e_{n+1}+B_ne_n+C_ne_{n-1},\ n\geq 0, \qquad (4.5)$$

($e_{-1}=0$) and continuous linear extension, and for this operator, $\|L\|\leq M$. The matrix of L relative to $\{e_n\}$ is J. Also, from (1.1) with L in the place of x, and (4.5), it easily follows (see [4]) that

$$P_n(L)e_0=e_n,\ n\geq 0 \qquad (4.6)$$

Now, Cramer's formula for the inverse of a matrix and simple calculations yield

$$\begin{aligned}((zI_n-J_n)^{-1}\overline{e}_0;\overline{e}_0)&=((z-L_n)^{-1}e_0;e_0)\\ &=\frac{P_n^*(z)}{P_n(z)},\ |z|>M,\end{aligned} \qquad (4.7)$$

where $\overline{e}_0=(1,0,\ldots,0)\in\mathbf{C}^n$ and L_n is the operator of l_2 whose matrix relative to $\{e_n\}$ is $\tilde{J}_n$. The operator L_n is bounded with $\|L_n\|\leq M$, and coincides with L on the span of $e_0,\ldots,e_{n-2}$. This implies that $L_{n+1}^ke_0=L^ke_0$, $k=0,1,\ldots,n-1$.

For any operator T of l_2 we write $(z-T)^{-1}=R(T,z)$. Then

Lemma 4.1. *For $|z|>M$,*

$$\lim_{n\to\infty}\|(R(L,z)-R(\tilde{L}_n,z))e_0\|=0, \qquad (4.8)$$

and the limit is uniform on $|z|\geq M'>M$.

Proof. For $|z|\geq M$ ([19], Chap.VIII),

$$R(L,z)=\sum_{k=0}^{\infty}\frac{L^k}{z^{k+1}},\quad R(\tilde{L}_n,z)=\sum_{k=0}^{\infty}\frac{L_n^k}{z^{k+1}}, \qquad (4.9)$$

the convergence of the series being in norm. Since $L_{n+1}^ke_0=L^ke_0$ for $k=0,1,\ldots,n-1$, then

$$(R(L,z))-R(\tilde{L}_{n+1},z))e_0=\sum_{k=n}^{\infty}\frac{L^ke_0}{z^{k+1}}-\frac{L_{n+1}^ke_0}{z^{k+1}}.$$

Thus, taking into account that $\|L\|\leq M$ and $\|\tilde{L}_{n+1}\|\leq M$, it follows that if $M'>M$ then

$$\begin{aligned}&\sup_{|z|>M'}\|(R(L,z))-R(\tilde{L}_{n+1},z))e_0\|\\ &\qquad\leq\frac{2}{M'}\sum_{k=n}^{\infty}\left(\frac{M}{M'}\right)^k,\end{aligned}$$

and, since the series on the right hand side is convergent, the assertion follows. □

Theorem 4.1. *If $X(z)=(R(L,z)e_0;e_0)$, then*

$$X(z)=\lim_{n\to\infty}\frac{P_n^*(z)}{P_n(z)},\ |z|>M, \qquad (4.10)$$

the limit is uniform on $|z|\geq M'$ for $M'>M$, and the function $X(z)$ is the limit of the continued fraction (1.7) and is an analytic function on $|z|>M$. Furthermore, if $\mathcal{L}$ is the moment functional of $\{P_n(x)\}$, then

$$\mathcal{L}(P(x))=\frac{1}{2\pi i}\int_C X(z)P(z)\,dz, \qquad (4.11)$$

where C is any positively oriented closed contour of $|z|>M$ around $z=0$.

Proof. That (4.10) holds follows at once from Lemma 4.1 and relation (4.7), and the analiticity of $X(z)$, from the uniform convergence of $\frac{P_n^*(z)}{P_n(z)}$ on $|z|\geq M'>M$. Since $\frac{P_n^*(z)}{P_n(z)}$ is, when (1.2) holds, the n-th convergent of (1.7) ([11], Chap.III), $X(z)$ is in fact the limit of (1.7). All that remains to be proved is relation (4.11). To do so we recall the Cauchy-Dunford representation ([19], Chap.VIII)

$$P(L)=\frac{1}{2\pi i}\int_C R(L,z)P(z)\,dz, \qquad (4.12)$$

which holds for any polynomial $P(x)$ and any positively oriented contour C of $|z|>\|L\|$ enclosing $z=0$, so that

$$(P(L)e_0;e_0)=\frac{1}{2\pi i}\int_C X(z)P(z)\,dz. \qquad (4.13)$$

Then, since $(P_n(L)e_0;e_0)=(e_n;e_0)=\delta_{n0}$, the assertion follows. □

We can easily prove that

$$\lim_{z\to\infty} zX(z) = 1 \qquad (4.14)$$

In fact, from (4.9) and Theorem 4.1,

$$X(z) = \sum_{n=0}^{\infty} \frac{(L^n e_0; e_0)}{z^{n+1}}, \ |z| > M,$$

so that

$$|zX(z) - 1| \leq \sum_{n=1}^{\infty} \frac{M^n}{|z|^n} = \frac{M/|z|}{1 - M/|z|} \to 0, \quad z \to \infty.$$

Relation (4.14) is, of course, well known, and will be used in the following argument. We observe that if $F(z)$ is analytic for $|z| > M' > 0$, $\lim_{z\to\infty} F(z) = 0$, and

$$\mathcal{L}(P(z)) = \frac{1}{2\pi i} \int_C F(z)P(z)\,dz$$

for any positively oriented contour $|z| > M'$ enclosing $z = 0$, necessarily $F(z) = X(z)$ for $|z| > \min\{M', M\}$. In fact, for any contour C in $|z| > \max\{M', M\}$,

$$\int_C (F(z) - X(z))\,P(z)\,dz = 0$$

for any polynomial $P(x)$. Hence, if $\sum_{n=-\infty}^{\infty} c_n z^n$ is the Laurent development of $F(z) - X(z)$ in $|z| > \max\{M', M\}$ then

$$c_{-n-1} = \frac{1}{2\pi i} \int_C (F(z) - X(z)) z^n\,dz = 0, \ n \geq 0,$$

so that $F(z) - X(z)$ is an entire function. Since $\lim_{z\to\infty}(F(z) - X(z)) = 0$, Liouville's Theorem implies that this function must vanish. This result has proved useful in [8], [10].

Remark 4.1. The connection between the supports of the distributions representing L and the spectrum of the associated matrix (4.1), an interesting aspect of the whole subject, is presently under research.

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