THERMODYNAMICAL ANALYSIS OF A FLUID MODEL FOR THE ONE PARTICLE SCHRÖDINGER EQUATION IN THE CONTEXT OF STOCHASTIC ELECTRODYNAMICS [1]

Por *Alfonso Rueda* [2]
and *A. Lecompte* [3]

ABSTRACT

We present here a thermodynamical analysis of the properties of the recently proposed probabilistic fluid model of Cavalleri and the way it leads to Schrödinger's equation in the contex of Stochastic Electrodynamics (SED). The assumptions required for the derivation are clearly stated and physically justified in the sense of thermodynamics. No rigorous proof, in the context of SED, of Schrödinger's equation is accomplished yet, since the statistical connection between the required assumptions and SED is not rigorously established. The analysis also touches on previous probabilistic fluid analogies not based on SED due to Santos, Nelson and others. The analysis finally shows the close underlying relationship between SED and ordinary quantum theory.

INTRODUCTION

Our principal aim is to carefully examine a recent proposal[1] for obtaining the Schrödinger equation in the context of classical physics. In this article this new attempt at obtaining Schrödinger's equation without the explicit or implicit introduction of a quantum assumption is examined. Using ideas of previous authors, the Schrödinger equation is here derived by taking advantage of the fluid analogy and within an entirely classical perspective, but this derivation is still far from rigorous. This derivation is an outgrow of those of Madelung[2], Bohm[3], Nelson[4], Santos[5], de la Peña-Auerbach and Cetto[6], and Cavalleri[1]. The analysis of the title relies on some ideas recently expounded in particular by Cavalleri[1]. A historical account of several previous attempts at deriving the Schrödinger's equation from classical physics is presented in the book by Jammer[7] but Jammer only covers work prior to 1974 and he does not cover Stochastic Electrodynamics (SED). Our derivation is given using the viewpoint of SED (also called Random Electrodynamics or Classical Electrodynamics with a Classical Electromagnetic Zero-Point Radiation).

SED is a new classical theory where the concept of a real zero-point field (ZPF) is introduced as an assumption. The reasonableness of such an assumption can be seen from the following. Consider an experiment involving a charged particle in the ordinary sense of Classical Electrodynamics (CE). In CE it is usual to think of the experiment as one where the particle is interacting with some specific set of fields, but where the experiment itself can be isolated from interaction with the rest of the Universe. Rigorously, however, for an experiment in microphysics involving very small particles the motion and acceleration of charges in the rest of the Universe should affect the particle, perhaps dramatically. The incoherent motion of all

1 Work partially supported by Colciencias.

2 Present address: Departamento de Física, Colegio Universitario de Humacao, CUH Station, Humacao, PR 00661.

3 Present address: Departament of Mathemathics. Universidad de los Andes, Apartado Aéreo 43116, Bogotá, Colombia, S.A.

the other charges in the Universe should exhibit itself in the form of a random e.m. field that is homogeneously and isotropically distributed. Furthermore, if according to special relativity, we think of all inertial reference frames as equivalent, then we should expect that such a random field is the same when looked at from different inertial frames of reference, i.e., the statistical properties of the ZPF random radiation are Lorentz invariant[8,9]. Mathematically the introduction of the ZPF random radiation comes very naturally. In ordinary CE the homogeneous boundary solutions to Maxwell's equations are assumed to be zero. In this new theory where all systems are regarded as open, on the contrary, the homogeneous boundary solution cannot be zero but rather corresponds to the random incoming radiation that adds itself to the other given fields that influence the experimental particle. As the random e.m. radiation is homogeneous and isotropic and looks the same from every inertial frame of reference, it has been shown that it has an energy density spectrum that behaves as $\omega^3\ d\omega$[2,8,9] We thus have for the ZPF energy density spectrum

$$\rho(\omega)d\omega = \frac{\hbar\omega^3}{2\pi^2 c^3}\ d\omega \qquad (1)$$

Hende, the same distribution as the ZPF of quantum theory results but now Planck's constant, $\hbar$, does not enter as the result of some quantization procedure but simply as a convenient constant that fixes the scale of the radiation and hence of the ramdom e.m. field (for a short and thorough review of this new theory see a review by Boyer[10] and references therein).

Due to severe mathematical difficulties the pace of progress in SED has not been rapid. There are only a few solid results. Nevertheless, strong indications that solutions to several problems can be found within SED are rather abundant as is the case for the hydrogen atom problem where, though no satisfactory solution has been found[11], there are some indications that SED may eventually present a satisfactory solution[12]. For a concise account of most relevant and pertinent SED results we refer the reader to one of our previous works[13]. The reasonableness of the statement that a satisfactory derivation of the one particle Schrödinger equation may soon be given classically and within the viewpoint of SED is shown in what follows.

Besides ideas from SED we make in the present work some assumptions about the dynamical behaviour of the probability density $\rho\ (\bar{r}, t)$ associated with the particle. These assumptions are consistent with the point of view of SED and are classical in their content. This possibly disproves claims that classical physics can never lead to an interpretation of Quantum Mechanics (QM). It may even be that further work will show that variants of SED may lead to the Schrödinger's equation. Such a possibility inescapably follows from the present derivation.

Our approach is phenomenological in the sense that no attempt is made at presenting a finished statistical analysis of the dynamical behaviour of a classical particle submitted both to a given well defined potential and to the classical electromagnetic ZPF. We let everything remain at a thermodynamic level of description. The contents of the paper are as follows. In next Section (Section 2) we present and discuss the basic physical assumptions from which we can later proceed in Section 4 to the construction of the Lagrangian density whose variation as proposed by Santos [5] and Cavalleri[1] leads via a Bernoulli equation to the Schrödinger equation. A discussion of these results emphasizing both the advantages and the limitation of the thermodynamic approach is presented in Section 5. Previously in Section 3 we show some important aspects of the diffusion of probability that we need for the proof in Section 4. There are also two Appendixes: Appendix A summarizes the SED arguments and calculations that lead to the frictionless or viscousless and hence irrotational flow of the probability fluid; Appendix B deals with some thermodynamic considerations on the probability flow that are required for completeness of presentation.

ASSUMPTIONS AND THEIR PHYSICAL JUSTIFICATION

We consider a single particle whose position we denote by $\bar{r}$. The particle is under the action of a potencial $\sqrt{(\bar{r})}$ that is well defined everywhere. The system is not closed. The action of all the other particles in the Universe is manifested by some random electromagnetic radiation independent of temperature called the background electromagnetic ZPF as previously explained[13]. The position of the particle is not known with precision but we can define a probability density associated with the position $\bar{r}$ of the particle at any time t, $\rho\ (\bar{r}, t)$.

The question of the motion of a single particle can be replaced with profit by the artifice of introducing an ensemble of identically prepared mutually moninteracting independent particles. Such an ensemble may be looked at as a fictitious gas or fluid whose mass density is $m\rho(\bar{r}, t)$. Every particle of the fluid should follow one of the possible paths of the actual particle. This fluid analogy will prove highly useful but it is of course not free from dangers. If carried uncritically it leads to spurious results. The probabilistic or statistical fluid here expounded has been known for a long time[2,4,5]. A flow of probability fluid of this sort was first introduced by Madelung[2] in 1922. This idea has been used more recently by Nelson[4], Santos[5] and Cava-

lleri[1]. The derivations of Nelson and Santos were not based on SED. Nelson's work was an outgrowth of previous ideas of Fényes[14]. Both Nelson's and Fényes' approaches constitute what is known today as Stochastic Mechanics (SM). The concept of a probabilistic fluid was briefly introduced in the work of Nelson more as helpful, visual and intuitive aid than as an essential tool. Not much later Santos[5] within the same framework of SM exploited very heavily the fluid analogy. But it was first Cavalleri[1] who introduced the probability fluid analogy within the context of SED in order to obtain the Schröndinger's equation. We follow and examine Cavalleri's inspiration in the present article by performing a thermodynamic analysis of his approach to the probability fluid, this analysis is performed in the context of SED. We explore the underlying assumptions required for obtaining the Schrödinger's equation and their connection with SED. In order to properly perform this job we clearly state and examine the required assumptions.

i) *Conservation of probability.* As the particle is neither created nor destroyed, it should be located somewhere in space and hence globally there is a conservation of probability that we can simply express by the normalization condition*.

$$\int \rho(\bar{r},t)\, d^3 r = I\,. \tag{2}$$

Actually we can demand more. We can demand probability to be locally conserved. This is expressed by the equation of continuity

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\,(\rho \bar{v}) = O\,. \tag{3}$$

where $\bar{v} = \bar{v}(\bar{r},t)$ is the velocity associated with the probability density at a given point $\bar{r}$ and time t. Using the terminology of Nelson we call $\bar{v}$ the *current velocity* of the particle. The definition of $\bar{v}$ can quantitavely be given if we realize that the ensemble of particles defines a probability density distribution in phase space $G\,(\bar{r},\,\bar{w},\,t)$ such that

$$\rho(\bar{r},t) = \int \sigma\,(\bar{r},\,\bar{w},\,t)\, d^3 w\,, \tag{4}$$

$$\bar{v}\,(\bar{r},\,t) = \int \bar{w}\,\sigma\,(\,\bar{r},\,\bar{w},\,t\,)\, d^3 w. \tag{5}$$

We will not make any further reference to the distribution σ in phase space. From now on our will proceed in configuration space.

ii). *Energy balance, time reversibility and isentropic evolution.* A charged particle that is under the influence of an accelerating force radiates energy at a rate given by Larmor's formula.

$$P\,rad = \frac{2e^2}{3c^3}\,(\ddot{\bar{r}})^2. \tag{6}$$

A particle under the influence of the ZPF may radiate energy in such a manner that the power radiated out P_{rad} and the power absorbed in $Pabs$ from the ZPF do not necessarily balance. In general we should not expect a balance to occur. For example, an ordinary monopolar particle* or an ordinary polarizable particle under the influence of the ZPF, but otherwise perfectly free and moninteracting, perform a random walk in velocity space to ever increasing translational kinetic energies[9,13,15-17]. In such a case obviously the power radiated and the power absorbed do not balance and instead we have that

$$\langle P_{abs} \rangle \;>\; \langle P_{rad} \rangle\,. \tag{7}$$

This cannot always be the case, of course. In a particle that is bound by a potential, as for example by the atomic Coulomb potential, we must expect that in equilibrium

$$\langle\, P_{abs}\,\rangle = \langle P_{rad} \rangle\,. \tag{8}$$

This is also the case for particles that collide often among themselves or with the walls of a cavity as well as for particles whose center of charge $\bar{x}$ and whose center of mass $\bar{X}$ do not coincide but where both centers together are bound by some kind of atractive force, provided the ZPF induced motion of the center of charge $\bar{x}$ around the center of mass $\bar{X}$ proceeds relativistically in a manner reminescent of the electron Zitterbewegung[18].

For a particle placed at time $t = 0$ in a potential $V(\bar{r})$ the balance of (8) does not occur instantaneously; only after a while will such a balance take place. The system relaxes after a very short but non zero time to the *quantum mechanical regime* as denoted in the terminology of de la Peña-Auerbach and Cetto[19]. The quantum mechanical regime is characterized by the energy balance of eq. (8).

In our derivation below, we must assume that the particle is in the quantum mechanical regime and hence that on the average there is no net loss or gain of energy. Now, from our knowledge of a single particle in a well defined potential, there is no degradation of the energy, i.e., the evolution is isentropic. This should manifest itself in the time reversibility of the single particle evolution of the probability density under the given potential. Thus, the quantum mechanical regime should display *time reversibility* in its corresponding evolution equations. Globally, this required isentropicity may at the outset be expressed by[5]

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* Note: that this automatically implies low-energy processes: particle creation is not considered to be possible.

* Provided we dismiss Zitterbewegung[18].

$$\frac{d}{dt}\int -\rho(\bar{r}, t)\log\,[\,\rho(\bar{r}, t)\,]d^3r = 0 \qquad (9)$$

or a similar expression involving $\rho(\bar{r}, \bar{w}, t)$ and integration over d^3w and d^3r.

However, for our purposes this global conservation of entropy, as was the case with the conservation of probability of eq. (2), is not enough. We also require here local conservation of entropy in the probability density flow, namely [20]

$$\frac{ds}{dt} = 0 \qquad (10)$$

where s is the entropy per unit mass and the total derivative with respect to time may be explicitely written as [20]

$$\frac{\partial s}{\partial t} + (\bar{v}\cdot\bar{\nabla})s = 0. \qquad (11)$$

This is the familiar equation describing adiabatic motion in a fluid. There is also an associated equation for the continuity of flow of entropy, that has the form[20]

$$\frac{\partial(\rho s)}{\partial t} + \bar{\nabla}(\rho s\bar{v}) = 0. \qquad (12)$$

Eqs. (11) and (12) are true in isentropic flow when there are no sources of entropy. Eqs. (10) to (12) are more demanding than (9) and imply it. The implication of course does not go in the reverse sense, but it is the concept of the local isentropicity of (10) to (12) that is required in the derivation.

iii). *Irrotational Flow.* As first shown by Einstein and Hopf[21], to first order in v/c, a polarizable particle that moves through homogeneous and isotropic random radiation experiences a frictional force of the form $-R\,\bar{v}$. However, if the radiation is the zero-point electromagnetic universal background radiation, because of its Lorentz invariant energy density spectrum, it remains homogeneous and isotropic in all inertial frames[8,9]. Hence, no electromagnetic friction should be expected to occur in that specific case. It has been shown that to first order in v/c this is indeed the case[9]. The simple proof that R = 0, when T = 0 and there is only the ZPF left,[16] carried out to all orders in v/c and for the case of polarizable particles is sketched in Appendix A. This extends thus to all orders in v/c the previously mentioned result (valid only to first order in v/c). For the convenience of the reader, in the case of monopolar particles, a recent proof that no friction is present to all orders in v/c is briefly mentioned in Appendix A. The reason for assuming a frictionless fluid here is this particular ZPF property rather than the fact that in an ensemble of noninteracting particles there is no inter-frictional effect among the particles because of their lack of mutual interactions[22]. The friction with random radiation is what causes the friction in our case. In the quantum mechanical regime, then, there is on the average a balance between the energy emitted and the energy absorbed from the ZPF, and if the random radiation is only the ZPF, there are no ZPF-induced frictional forces. This is so not only when there are no other forces than those due to the ZPF itself but also when the particle moves through a given potential (plus the ZPF).

The Einstein and Hopf loss of energy[21], due to the frictional force for a freely moving particle through random radiation, should not be confused with the instantaneous radiation reaction force that appears in the ordinary Abraham-Lorentz equation:

$$m\ddot{\bar{r}} = \bar{F} + m\Gamma\dddot{\bar{r}}\;;\; \bar{F} = -m\bar{\nabla}V \qquad (13)$$

which we write without the random radiation term. m is the mass of the particle and $\bar{F}$ is the force caused by the potential V $(\bar{r})$ which for convenience we define here per unit mass. The time constant

$$\Gamma = \frac{2e^2}{3mc^3}$$

for the case of an electron has a very small value of approximately 10^{-23} s. In order to avoid runaway solutions it is costumary to write [23]

$$m\ddot{\bar{r}} = \Gamma^{-1}\int_t^{\infty} e^{(t-t')/\Gamma}\,\bar{F}(\bar{r}(t'))dt' \qquad (14)$$

and hence

$$\Gamma m\dddot{\bar{r}} = \frac{1}{\Gamma}\int_t^{\infty} e^{(t-t')/\Gamma}\,\bar{F}(\bar{r}(t'))\,dt' -$$

$$-\bar{F}(\bar{r}(t)) = m\ddot{\bar{r}} - \bar{F} \qquad (15)$$

from which we recover the Abraham-Lorentz eq. (13), but no runaway solutions appear in (14). It is usually useful to approximate the integro-differential equation (14) by a simple differential equation. We expand $\bar{F}$ in powers of the small quantity Γ and, keeping the first two terms of the expansion, obtain

$$\bar{F}(\bar{r}(t')) = \bar{F}(\bar{r}(t)) + (t'-t)[\dot{\bar{r}}(t')\cdot\bar{\nabla}\bar{F}] + \ldots \qquad (t'-t) = 0 \qquad (16)$$

Replacing (16) in (14) we get

$$m\ddot{\bar{r}} \doteq \bar{F}(\bar{r}(t)) + (\dot{\bar{r}}(t)\cdot\bar{\nabla})\bar{F}(\bar{r}(t)) \qquad (17)$$

which in most cases is a good approximation to (16). In the presence of random radiation, besides the potential force in (14) we should include the force due to the random field. Thus the velocity dependent force of (17) should come *in addition* to the possible frictional force caused by random radiaton. The velocity dependent force of (17) is just another way of writing the radiation reaction force when $\bar{F} \neq 0$. Such a force is always present and in general its nature is different from that of the Einstein-Hopf frictional force for motion through random radiation. The potential force $\bar{F}$ does not cancel for the case of motion throught the ZPF as is the case for the Einstein-Hopf frictional force. Moreover, in the case of a particle moving under the potential $V(\bar{r})$ and through the ZPF, on the average the energy balance of eq. (8) is strictly obeyed as it should be in the quantum mechanical regime. This shows that the effect of the radiation reaction force in eqs. (13) and (17) is on the average compensated by the power absorbed from the ZPF as explained by de la Peña and Cetto[19]. We thus have that no frictional forces may appear for a single particle when, being in the quantum mechanical regime, it moves under a well-defined potential and through the ZPF. Hence, the associated probability distribution $\rho(\bar{r}, t)$ should evolve in a characteristic frictionless manner. This implies that the associated probabilistic fluid analogy should display no friction, i.e., within the point of view of the analogy with a real fluid, the probabilistic fluid should be without viscosity.

Invoking the fluid analogy we can bring into account Thomson's theorem which for a fluid without viscosity essentially implies that its flow is irrotational[24]. We therefore should expect that

$$\bar{\nabla} \times \bar{v} = 0 \tag{18}$$

iv). *Brownian motion and diffusion equation.* A particle which at a given point in time t_0 has a well defined position $\bar{r}_0$, executes two kinds of motions. First, in the absence of the ZPF it follows its classical trajectory determined by the initial velocity $\bar{v}(t.)$ at time $t = t_0$ and by the potential $V(\bar{r})$. The presence of the ZPF introduces the second kind of motion. Superimposed on the first there is a random fluctuating motion due to the ZPF. We may view this second kind of motion as constituted by a whole ensemble of particles that move along all kinds of possible trajectories (see beginning of this Section). We can thus conceive the motion under the ZPF as some sort of Brownian motion. It differs from ordinary diffusion under random radiation in that in the present case the radiation displays a spectrum that is not constant - with ω - i.e., it displays a spectrum that is not white. Mathematically then the pertinent stochastic process is non-Markovian. This is an important point to which we briefly come back again in Section 5. If we consider, however, the randomizing influence of the ZPF as producing some sort of Browninan motion and omit for the moment the presence of the potential $V(\bar{r})$ we can assume that in the quantum mechanical regime the evolution of the ensemble may be represented by a stochastic process that approximately behaves as a Wiener-Einstein process. It is well known that for the Wiener-Einstein process the probability density obeys a diffusion equation[25]

$$\frac{\partial \rho}{\partial t} = D \nabla^2 \rho . \tag{19}$$

This last equation is strictly valid when we place ourselves in an inertial frame that instantaneously moves with the particle, $\bar{v} = 0$, and there are no other strong forces besides the ZPF acting on the particle. If that is not the case and there are strong external forces applied to the particle besides that due to the ZPF, then (19) should be replaced by the appropriate Fokker-Planck equation.

Following Santos[5] we construct in Section 4 an appropiate Lagrangian for the ensemble taking advantage of the probabilistic fluid concept[1].

v). *Local force density and dynamical interaction among adjacent probability elements.* This is one of the most distinctive assumptions of the fluid analogy. It refers to the local dynamics of each individual probability element. The assumption may be considered to consist of two parts. First we have that, as in all fluids, there are local bulk forces that are applied externally to the fluid by means of various kinds of fields. The total force $\bar{F}$ applied to our particle is thus the superposition of the forces applied to all and each one of the different volume elements of the probability fluid. We write

$$\bar{F} = \int \frac{\bar{f}(\bar{r}, t)}{m} \rho(\bar{r}, t)\, d^3 r \tag{20}$$

where $\bar{f}(\bar{r}, t)/m$ is thus the local applied force density per unit mass of probability fluid. The second part of the assumption refers to the mutual interaction among neighboring or adjacent probability elements; i.e., as in ordinary fluids, we assume that adjacent volume elements of the probability fluid can mutually interact, and apply mutual pressures.

As a mass $m\, \rho(\bar{r}, t)\, dV$ is associated with a given probability element it is easy to see that the probability elements, in analogous fashion to ordinary fluids, should obey Newton's laws which for the fluid element at point r and time t yield

$$m\, \rho(\bar{r}, t)\, d^3 r \frac{d\bar{v}}{dt} = [\rho(\bar{r},t)\, \bar{f}(\bar{r},t) - \bar{\nabla} p(\bar{r},t)] d^3 r \tag{21}$$

By $p(\bar{r}, t)$ we denote the pressure at $(\bar{r}, t)$. In our case this pressure may also be negative, i.e., the neighboring probability elements may mutually apply a tensive force instead of the more familiar comprensive force. However, as pointed out at the end of assumption (iii) the probability fluid has no viscosity and hence our stress tensor is purely diagonal and no shear forces in the fluid may appear as depicted in (21), where shear forces are absent.

By realizing that

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla})\bar{v} \tag{22}$$

and using the well-known vector identity

$$(\bar{v} \cdot \bar{\nabla})\bar{v} = \bar{\nabla}(\tfrac{1}{2} v^2) - \bar{v} \times (\bar{\nabla} \times \bar{v}),$$

Newton's second law of eq. (21) may be written

$$\frac{\partial \bar{v}}{\partial t} + \bar{\nabla}(\tfrac{1}{2} v^2) + \frac{1}{m\rho} \bar{\nabla} \rho - \bar{f} = 0 \tag{23}$$

where we already made use of the irrotationality condition of eq. (18). Further transformations of eq. (23) will be given in Section 4 and in Appendix B.

DIFUSION VELOCITY AND DIFFUSIVE KINETIC–ENERGY

The diffusion of particles in the ensemble or in the fictitious fluid is characterized by two processes: a) *mixing*, i. e., the mutual internal exchange of relative positions among the particles of the fluid, and b) *macroscopi displacement of the fluid in a sense opposite to the concentration gradient*. Using the terminology of Boltzmann, the kinetic energy of the mixing motion may be classified as part of the microscopic kinetic energy. There are also some other components of the microscopic kinetic energy as are the ZPF induced vibrations of the components of a composite (polarizable) particle or the ZPF induced Zitterbewegung which is a relativistic oscillation of the center of charge around the center of mass described in more detail elsewhere[18]. (See also Section 2 (ii).)

It is not hard to see that only the kinetic energy of b) above is relevant in the variations of the Lagrangian. The variations of the Lagrangian that are of interest here refer only to the macroscopic degrees of freedom of the probability fluid. The corresponding concentration or diffusion velocity of b) can be found as follows. Let us compare (19) with (3). We should have that

$$\bar{\nabla}(\rho \bar{v} + D \bar{\nabla} \rho) = 0, \tag{24}$$

and by integration we obtain then

$$\rho \bar{v} = \rho (\bar{w} - D \frac{\bar{\nabla} \rho}{\rho}). \tag{25}$$

We define

$$\bar{u} = - D \frac{\bar{\nabla} \rho}{\rho} \tag{26}$$

as the *diffusion* or *concentration velocity*. *Our* $\bar{u}$ is actually the negative of the osmotic velocity introduced by Nelson[4]. The velocity $\bar{w}$ is such that the current associated with it is divergenceless, i.e., it is the component of the current velocity $\bar{v}$ that not related to net flux out of a fixed comoving element of volumen dV. We then have

$$\bar{\nabla} \cdot (\rho \bar{w}) = 0 \tag{27}$$

and

$$\bar{v} = \bar{w} + \bar{\mu}. \tag{28}$$

The interesting thing here is that the kinetic energy density of b above, i.e., that associated with $\bar{v}$, namely

$$K_D = \frac{1}{2} \rho m (- D \frac{\bar{\nabla} \rho}{\rho})^2 = \frac{1}{2} m D^2 \frac{(\bar{\nabla} \rho)^2}{\rho}, \tag{29}$$

and not that associated with the mixing of a) above, is the one that is relevant for the Lagrangian. This point has been discussed by Santos[5] and Cavalleri[1]. We will come back to it in next section.

LAGRANGIAN DENSITY BERNOULLI EQUATION AND SCHRÖDINGER'S EQUATION

From equation (18) we can see that the velocity $\bar{v}$ can be writtenin the form

$$\bar{v} = -\bar{\nabla} \Phi \tag{30}$$

where $\Phi(\bar{r}, t)$ is the velocity potential. The fluid analogy proves to be particularly useful since all the basic equations and the formalism of irrotational isentropic fluids[24] can be applied to the probabilistic fluid of our problem. Indeed, we give in Appendix B the well known thermodynamic relationship for irrotational isentropic fluids as applied to the present case. The last paragraph of Appendix B leads to

$$\mathcal{L} = m\rho \left[\frac{\partial \Phi}{\partial t} - \tfrac{1}{2} (\bar{\nabla} \phi)^2 - V - \epsilon \right] \tag{31}$$

where V is the potential energy per unit mass defined in eq. (13) and ϵ is the internal energy per unit mass, $\epsilon = \epsilon(\rho,s)$. When (31) is varied with respect to Φ we obtain the Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\Phi/\partial t)} + \bar{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial(\bar{\nabla}\Phi} \tag{32}$$

which immediately gives

$$\frac{\partial \rho}{\partial t} - \bar{\nabla}\cdot(\rho\bar{\nabla}\Phi) = 0 \tag{33}$$

from which we can recognize the equation of continuity (3). The other Lagrange equation results when we consider the variation with respecto to ρ. It is

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\rho/\partial t)} + \bar{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial(\bar{\nabla}\rho)} . \tag{34}$$

Before explicitely performing this variation with respecto to ρ we should specify better the nature of the internal energy ϵ of eq. (31).In (32) and (33) we could forget about this specification since the internal energy ϵ obviously did not depend on the velocity potential. In order to specifiy ϵ we recall that in numeral v) of section 2 we briefly discussed the problem of the internal energy. From that discussion it followed that

$$m\rho\epsilon = \tfrac{1}{2}m\rho\left(D\frac{\bar{\nabla}\rho}{\rho}\right)^2 + m\rho\,\epsilon_i \tag{35}$$

where ϵ_i depends on internal parameters of the particle as discussed in v). Thus ϵ_i is a constant that may be absorbed in the potential V as far as variations with respecto to ρ are concerned. With this in mind we thus write[1,2,4,5]

$$\epsilon = \tfrac{1}{2}\left(D\frac{\bar{\nabla}\rho}{\rho}\right)^2 \tag{36}$$

Introducing (36) in (31) and performing the operations of (34) we are left with

$$\frac{\partial\Phi}{\partial t} - \tfrac{1}{2}(\bar{\nabla}\Phi)^2 - V - D^2\left[\tfrac{1}{2}\left(\frac{\bar{\nabla}\rho}{\rho}\right)^2 - \frac{\bar{\nabla}^2\rho}{\rho}\right] \tag{37}$$

This is just the Bernoulli's equation for isentropic irrational flow of (B10) and (B7). In order to see this we only need to identify the pressure p with the expression

$$p = -m D^2 \nabla^2 \rho . \tag{38}$$

and we may write (37) as

$$\epsilon + \frac{p}{\rho m} + V + \tfrac{1}{2}(\bar{\nabla}\Phi)^2 - \frac{\partial\Phi}{\partial t} = 0 \tag{39}$$

which is the usual expression for the Bernoulli equation for an irrotational isentropic fluid[24] of mass density ρm as can be seen from (B10) and (B7). The Bernoulli equation expresses conservation of energy. This interesting fact is consistent with the assumpton (Section 2 assumption ii) that in the quantum mechanical regime there is conservation of energy on the average. Besides expressing conservation of energy, the isentropocity and the irrotationality of assumptions (ii) and (iv) are also taken care of in (37).

Equation (37) was first discovered by Bohm long ago[3]. He, however, did not associate it with any kind of fluid model. Bohm was the first to introduce the transformation[3]

$$\psi(\bar{r},t) = \rho^{1/2}\exp\left(-\frac{2\Phi}{2D}\right) \tag{40}$$

which when replaced in eq. (37) yields the expression

$$iD\frac{\partial\psi}{\partial t} = -D^2\nabla^2 x + \frac{V}{2}\psi \tag{41}$$

and its complex conjugate. As suggested by Cavalleri[1] the diffusion constant D may be evaluated by searching for a problem solved classically by means of SED and comparing the corresponding solution with that derived from quantum theory. Applying (41) in one dimension to the harmonic oscillator problem we obtain $\langle x^2\rangle = D/w_o$, where w_o is the characteristic frequency. SED gives $\langle x^2\rangle = \frac{h}{2mw_o}$ + radiative corrections. We obtain then that with good approximation

$$D = \hbar/2m \tag{42}$$

which is the value for the diffusion constant of (19) that has to be adopted in SED in order to establish a correspondance with quantum theory. With this provission, (41) becomes the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}(\bar{r},t) = -\frac{\hbar^2}{2m}\bar{\nabla}^2\psi(\bar{r},t) + V(\bar{r})\psi(\bar{r},t) \tag{43}$$

DISCUSSION

The one-particle Schrödinger equation is all one may attempt to derive with the fluid model. In the present manuscript no attempt is made at exploring the many-particles Schrödinger equation where mutually interacting particles are considered. This requires further conceptualization an the simple fluid analogy here considered cannot by itself provide the satisfactory means for a derivation. Additional assumptions are certainly needed for the more complex many-particles cases. Moreover, it is possible that even for the two-particle case we have to strongly modify several of the essential aspects related to the idea of the fluid analogy.

For stationary states we know that the relevant underlying stochastic process must be Markovian. For nonstationary states of course that is not necessarily the case[25]. For nonstationary states we are at pains even to know if QM leads to a subjacent non-Markovian process[25]. In general, QM does not even seem to undisputably be associated with a stochastic process[26]. However, for stationary states and surely for the ground state, QM clearly shows that the underlying process must be a stochastic process of the Markovian type[25]. In assumption (iv) of Section 2 we have implicitly assumed that the underlying process that leads to the Schrödinger equation is a Markov-diffusion process, that obeys eq. (19). Such a point deserves to be clearly stated. We thus realize that the Markovian character of the distribution has to be introduced in an *ad hoc* manner because the fundamental equations of SED, like the Braffort Marshall equation[27], are not clearly related to a Markov process. It is obvious instead that SED in general is associated with a non-Markovian process. This can be seen from the peculiar form of the SED propagator[28]. The diffusion process introduced in Section 2, assumption (iv), is necessarily a Markov-diffusion process which rigourously can be derived from the basic equations of SED provided some sort of Markovian approximation is performed. Such an SED Markovian approximation or Markovian limit has been widely studied[6, 11, 29, 30]. So our procedure is not at variance with that of the tradicional wiew point of SED. Our analysis is thermodynamical and gives plausibility arguments based on general principles that exploit the well-known phenomenology of the microworld. We invoke principles like conservation of probability, time reversibility, etc.. However, we do not examine how the basic equations of SED lead to equations like our Bernoulli equation (39) which is based on those principles. Such a task is left for a more fundamental approach based on a statistical analysis.

Our analysis exploits a fluid analogy for the probability density evolution leading to the Schrödinger equation, which comes from nothing else than the result of applying the Bohm transformation to the Bernoulli equation. It is perfectly conceivable though, that the Schrödinger equation may be derived using the same equations here presented but without invoking any kind of fluid analogy. However, the fluid analogy greatly facilitates the work because of two reasons. First, it provides the well known mathematical tools, which in this case are all the hydrodynamical equations of irrotational, isentropic flow. Second, it helps the intuition by presenting the very familiar scenario of the behavior of fluids.

It is also interesting to point out that, in carrying out the fluid analogy, the ZPF disappears from the formulation. It is nowhere manifested in the Bernoulli equation and hence it cannot appear in Schrödinger's equation. This is due to the energy balancing that on the average one assumes to hold: on the average no energy enters or leaves the system

$$\langle P \rangle = \langle P_{abs} - P_{rad} \rangle = 0 \qquad (44)$$

once the system has already relaxed to the quantum mechanical regime. But instantaneously of course $P \neq 0$ in general. Hence, in the more refined statistical theory mentioned above an expression should appear that is more general than the energy balancing expression implicit in the Bernoulli equation. This has indeed been the case with the approach followed in the attempt of de la Peña-Auerbach and Cetto[19], who, after deriving a non-Markovian Fokker-Planck-type equation in phase space, construct from it an infinite hyerarchy of equations that for times long compared with an appropriately defined correlation time, approximately reduce just to two equations: the Bernoulli equation and the continuity equation. From these last two equations the Schrödinger equation follows by the usual Bohm transformation[3, 7]. A concrete example of the energy balance of eqs. (44) and (8) has been given by Boyer[12] who could show how even the isotropicity of the ZPF energy distribution is not destroyed by a dipole that is held firmly fixed at the origin and interacts with the ZPF. On the average the ZPF energy remains homogeneously and isotropically distributed in such a manner that the average value of the ZPF Poynting vector vanishes everywhere and in all directions, $\langle \vec{S}_{ZPF} \rangle = 0$ Hence in this particular example, we know that (8) and (44) are exactly obeyed.

It is interesting to point out that this energy balancing of eqs. (44) and (8) may be related to the appearance of nodes in the solutions of the Schrödinger's equation. These nodes are unexpected classically as can be seen from eqs. (4) and (5). Cavalleri[31] observes that the Schrödinger's equation represents an average behaviour consistent with the energy balancing $\langle P \rangle = 0$ implicit in the equation of Bernoulli as applied to the probability fluid. However, we know that instantaneously, in general $P \neq 0$ and no nodes should be expected to

appear for this more general case. The fluctuations of P were not taken into account in our derivation but indeed the variance of P should be different from zero, $\langle(P-\langle P\rangle)^2\rangle \neq 0$. The energy fluctuations allow for more general equations than Schrödinger's whose solutions should not yield an exactly null probability density separating two regions of positive probability, and hence the passing of the particle through the "node", which should be a region of low but not null probability density, is not impossible in SED. It is only less probable. It is characteristic of the fluid analogies for the flow of probability density that neighbouring probability elements interact among themselves in essentially identical manner as neighbouring elements of a fluid interact, i.e., two neighbouring elements can mutually apply pressure or tensive forces as occurs in ordinary fluids (no shear forces are allowed in the irrotational probability fluid). Moreover, each element reacts to the forces applied by its neighbouring elements in the same manner that a real element of fluid reacts. It is accelerated in an inverse relation to its mass as dictated by Newton's laws! Such a behavior is far from intuitively obvious for probability elements! A justification for this peculiar behavior perhaps is that by requiring this from each probability element the whole probability distribution and hence the particle is forced to obey Newton's laws as a whole. Due to the linearity of the differential equations of motion, superposition is obeyed for the whole system and all mutual forces among probability elements cancel leaving only the net externally applied force to accelerate the particle.

Last but not least we discuss some aspects of hypothesis (ii) where it is assumed that the system is in the quantum mechanical regime. In this case the particle submitted to the actions of the potential $V(\bar{r})$ and of the ZPF is in some state of equilibrium with the ZPF in such a way that the probability density evolves isentropically. This assumption is essential in order to guarantee the well known time reversibility property of the Schrödinger equation (42). We have boldly assumed that the entropy associated with the probability fluid remains constant, eqs. (9)-(12). The example of the fixed dipole that interacts with the ZPF explored by Boyer[12] makes us think that the entropy of the ZPF radiation remains constant also. Very little is known about the statistical thermodynamics of the ZPF in interaction with material systems. Boyer[32] has proposed that the ZPF represents radiation at the maximum degree of thermal degradation in the Universe, i.e., radiation with the maximum possible entropy. The thermodynamical questions related to the problem of the possible equilibrium of the ZPF with a gas of electromagnetically interacting unconfined particles have been posed elsewhere[33].

A rigorous analysis of the evolution of the entropy of radiation for the radiant ZPF energy seems highly nontrivial, not only because of the divergence of the energy density in (1) but also because the existing nonequilibrium entropy of radiation formalism, known since Planck's second theory[34], cannot be applied in the $T_{\nu k} \to 0$ limit as it refers exclusively to thermal radiation $(T_{\nu k} > 0)$. $T_{\nu k}$ is the temperature of the pencil of radiation of frequency ν moving in the $\bar{k}$-direction[35].

Note added in manuscript.- This article was presented in the Latinamerican School of Physics, Escuela Latinoamerican de Física ELAF-82 and a summary of its contents may be found in the corresponding Proceedings[36]. Further work along these lines[37] has shown that ordinary SED, i.e., SED as such without the recourse of any additional assumption, cannot lead to Schrödinger's equation because a free electromagnetically interacting particle is seen to increase its translational kinetic energy in ordinary SED[9, 13, 15]. This refers then to assumption (ii) which in this light is seen to depart somewhat from ordinary SED. Cavalleri has more recently produced a new proof[38] of Schrödinger's equation from SED but implemented with a non-electromagnetic form of Zitterbewegung as the additional assumption.

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APPENDIX A – NO EINSTEIN-HOPF FRICTIONAL FORCE FOR PARTICLES MOVING THROUGH THE ZPF. SIMPLE PROOF FOR POLARIZABLE PARTICLES TO ALL ORDERS IN ν/c

First we treat the case of polarizable particles and next we only very briefly comment on relevant aspects of the case of monopolar particles. The merit here is that our proof is good for relativistic speeds, i.e., to all orders in ν/c. Consider a polarizable particle of mass m and velocity $\bar{v}_t$ at time t submitted to the action of random radiation. If no walls confine the system of particle and radiation we write

$$m\bar{v}_{t+\tau} = m\bar{v}_t + \bar{I}\tau \tag{A.1}$$

where τ is a short time interval. The net impulse acquired during this time interval we denote by $\bar{I}\tau$. This impulse may be broken up into two separate parts

$$\bar{I}_\tau = \bar{\Delta} + \bar{P} \tag{A.2}$$

where $\bar{\Delta}$ represents a fluctuating part and $\bar{P}$ a velocity dependent drag, that we may call the systematic part. *To first order in* v/c it can be shown that for a dipole whose natural frequency is ω_o

$$\bar{P} \equiv \langle \bar{I}_T \rangle = - R_T \bar{v}_t \qquad (A.3)$$

with

$$R = 3\pi^2 \Gamma_{qc} \left[\rho(\omega, T) - \frac{1}{3} \omega \frac{\partial \rho(\omega, T)}{\partial \omega} \right] \qquad (A.4)$$

where

$$\Gamma_q = \frac{2q^2}{3Mc^3} . \qquad (A.5)$$

The proof of eqs. (A.3) and (A.4) which are well known relationships follow from the *general proof* we give below which is *valid to all orders in v/c.*

In (A.5) q represents the charge and M the mass of the entity that oscillates inside the dipolar particle. Observe that this mass M and this charge q do not necessarily coincide with the total particle charge e and with the particle proper mass, m, respectively, but that they may be quite different as previously pointed out[15]. The spectral energy density may refer for example to the thermal background (restricted Planck plus zero-point spectrum).

$$\rho(\omega, T) = \frac{\omega^2}{\pi^2 c^3} \left(\frac{\hbar \omega}{e^{\hbar\omega/kT} - 1} + \frac{\hbar \omega}{2} \right) \qquad (A.6)$$

which for $T = 0$ becomes the zero-point universal background radiation of (1).

From (A.2) and for the zero-point radiation of (A.5) it can be seen that to first order in v/c, we obtain $R = 0$. But rigorously this holds only to first order in v/c. However, due to the Lorentz invariance of the ZPF energy density spectrum the radiation remains isotropic in all possible inertial frames instantaneously at rest with respect to the particle. The extension of this result to arbitrarily large speeds looks very natural. However, a rigorous proof is required. It is given next.

Using standard notation we write the random electric and magnetic fields in the form[12]

$$\bar{E}(\bar{r}, t) = \sum_{\lambda=1}^{2} \int d^3k \, \hat{e}(\bar{k}, \lambda) \, h(\omega_{\bar{k}} T) \times$$

$$\times \cos(\omega_{\bar{k}} t - \bar{k} \cdot \bar{r} - \theta(\bar{k}, \lambda)) \qquad (A.7)$$

where $\omega_{\bar{k}} = ck$ and $\{\hat{e}(\bar{k}, 1), \hat{e}(\bar{k}, 2), \hat{k}\}$, $\hat{k} = \bar{k}/k$ form and orthonormal basis. The magnetic field $\bar{B}(\bar{r}, t)$ is obtained simply by replacing in eq. (A.7) the unit vector $\hat{e}(\bar{k}, \lambda)$ by the unit vector $\hat{k} \times \hat{e}$ $(\bar{k}, \lambda)$. The random character of the fields is introduced in the classical manner[7] of Planck and of Einstein and Hopf[21] by means of the random phase angle $\theta(\bar{k}, \lambda)$, $\lambda = h^2$. This angular function, with a uniform probability distribution takes an arbitrary value between 0 and 2π. The scale of the field is set by means of the amplitude $h(\omega_{\bar{k}}, T)$ in such a manner that the spectral energy density is related to $h(\omega, T)$ by

$$\rho(\omega, T) = \frac{\omega^2}{c^3} h^2(\omega, T) . \qquad (A.8)$$

Here we are considering the amplitude h as measured in the inertial frame of the cavity that contains the thermal radiation. Otherwise the thermal radiation would loose its isotropy and we should have to speak of $h[\bar{k}]$ because the amplitude would then of course depend also on the direction of the incoming waves. Beware of the fact that in the anisotropic case no unique temperature T can be assigned to the radiation as different pencils of radiation have different temperatures. For the case of the ZPF, however, the radiation looks isotropic from all inertial frames and hence we way write

$$h^2(\omega) = \frac{\hbar \omega}{2\pi^2} . \qquad (A.9)$$

Consider now a new reference frame S' moving with speed v along the positive x-direction with respect to the inertial frame S, which is the frame where the fields were originally defined by eqs. (A.1) to (A.3). Using the Lorentz transformation of the fields and the expressions

$$k_x = \gamma \left(k'_x + \frac{v\omega'}{c} \right); \; k_y = k'_y \; ; \; k_z = k'_z \qquad (A.10)$$

$$\omega = \gamma(\omega' + v k'_x) \; ; \; \gamma = (1 - v^2/c^2)^{-1/2} \qquad (A.11)$$

$$d^3k = d^3k' \, \gamma \left(1 + \frac{v k'_x}{c} \right) \qquad (A.12)$$

we can then write

$$\bar{E}'(\bar{r}', t') = \sum_{\lambda=1}^{2} \int d^3k \left\{ \epsilon_x i_x + \gamma \left(\epsilon_y - \frac{v}{c} (\hat{k} \times \hat{e})_z \right) i_y + \right.$$

$$\left. + \gamma \left(\epsilon_x + \frac{v}{c} (\hat{k} \times \hat{e})_y \right) i_z \right\} h(\omega_{\bar{k}})$$

$$\cos\left(\omega_{\bar{k}} t - \bar{k} \cdot \bar{r} - \theta(\bar{k}, \lambda) \right) \qquad (A.13)$$

where the Lorentz invariance of the phases has been used. For averaging over the random phases we introduce the orthogonality relations

$$\langle Cos\,(\theta(\bar{k}, \lambda_1))\, Cos\,(\theta(\bar{k}_2, \lambda_2))\rangle =$$

$$\langle Sin\,(\theta(\bar{k}, \lambda_1)\, Sin\,(\theta(\bar{k}_2, \lambda_2))\rangle =$$

$$\frac{1}{2}\delta\lambda_1\lambda_2\ \delta\,(\bar{k}_1 - \bar{k}_2) \qquad (A.14)$$

$$\langle Cos(\theta(\bar{k}_1, \lambda_1))\ Sin\,(\theta(\bar{k}_2, \lambda_2))\rangle = 0 \qquad (A.15)$$

and performing the required operation we obtain

$$\langle E'^2\rangle = \int h^2(\omega)\ \frac{d^3\,k'}{\gamma(1 + \frac{\bar{v}\cdot k^1}{\omega^1})} \qquad (A.16)$$

In the inertial frame S' of the particle, the cavity radiation does not look isotropic, We denote this fact by writing $h'\,[\,\bar{k}'\,]$. In the S'. frame the electric field is then

$$\bar{E}'(\bar{r}, t') = \sum_{\lambda=1}^{2} \int d^3\,k'\ h'\,[\bar{k}']\ \hat{\epsilon}(\bar{k}', \lambda')\ \times$$

$$\times\, Cos\,(\omega\bar{k}'\,t' - \bar{k}'\cdot\bar{r}' - \theta'\,(\bar{k}', \lambda')) \qquad (A.17)$$

which, after squaring and averaging with (A.14) and (A.15), gives

$$\langle E'^2\rangle = \int h'^2\ [\bar{k}']\, d^3\ k'\ . \qquad (A.18)$$

Comparing with eq. (A.16) we obtain

$$(h'\,[\bar{k}'])^2 = \frac{\left\{h\,(\omega'\gamma\left(1 + \frac{\bar{v}\cdot\bar{k}'}{\omega'}\right)\right\}^2}{\gamma\left(1 + \frac{\bar{v}\cdot\bar{k}'}{\omega'}\right)}\ . \qquad (A.19)$$

For the zero-point radiation due to the isotropy of the spectrum in all inertial frames the above expression simply becomes

$$h'^2\,[\bar{k}'] = h^2\,(\omega') = \frac{\hbar\omega'}{2\pi^2}\ . \qquad (A.20)$$

This easily confirms then that eq. (1) is the only spectrum that remains homogeneous and isotropic in all inertial frames. For thermal radiation though, (A.19) shows how the isotropy is lost when transforming away to a reference frame that moves with respect to the cavity walls. In the low velocity limit, $v, \ll c$, (A.19) gives

$$h'^2\,[\bar{k}'] \cong h^2(\omega') - \left[h^2\,(\omega') - \omega'\,\frac{\partial h^2\,(\omega')}{\partial\omega'}\right]\frac{\bar{v}\ \cdot\ \bar{k}'}{\omega'}\ . \qquad (A.21)$$

This expression displays the anisotropy, due to the particle motion through the field.

Consider the polarizable particle of (A.1). Denote its instantaneous dipole moment by $\bar{d}$, its rest mass by m and its velocity, as measured at time t with respect to the frame of reference fixed in the cavity, by $\bar{v}_t$. In the frame of reference S', where the particle at time t is instantaneously at rest we denote all quantities with a prime. The dipole moment obeys in this new frame an equation of the form[9,15]

$$\ddot{\bar{d}}' - \Gamma_q\,\dddot{\bar{d}}' + \omega_o^2\,\bar{d}' = \frac{3}{2}\Gamma q\ c^3\ \bar{E}' \qquad (A.22)$$

where ω_o is the natural frequency of resonance of the dipole and $\bar{E}'$ denotes the electric field in the particle's frame of reference S'. The Lorentz force on the particle is

$$m\ddot{\bar{r}}' = (\bar{d}'\cdot\bar{\nabla}')\,\bar{E}' + \frac{\dot{\bar{d}}'}{c}\times\bar{B}'\ . \qquad (A.23)$$

Next we solve for the dipole moment

$$\bar{P}'(t') = \sum_{\lambda=1}^{2}\int d^3k'\ f\,(\bar{k}')\ Cos\,(\omega' t' - \alpha(\omega')$$

$$- \theta'(\bar{k}',\ \lambda')\quad \hat{\epsilon}(\bar{k}', \lambda') \qquad (A.24)$$

where

$$f\,(\bar{k}') = \frac{3c^3}{2\omega'^3}\ Sin\,\alpha(\omega')\,h'\,[\bar{k}'] \qquad (A.25)$$

and

$$tg\,(\alpha(\omega')) = \frac{\Gamma_q\ \omega'^3}{\omega_o^2 - \omega'^2}\ . \qquad (A.26)$$

The dipole approximation may be used as discussed in a previous work[16].

The particle velocity $\dot{\bar{\pi}}'$ at time τ is

$$m\dot{\bar{r}}'(\tau) = \frac{1}{c}\,\bar{d}'\times\bar{B}\int_0^{\tau} + \int_6^{\tau}\bar{\nabla}'(\bar{d}'\cdot\bar{E}')\,dt' \qquad (A.27)$$

where τ denotes now a sufficiently short time interval *as measured in the inertial frame of reference S'*. For simplicity we have changed the time origin. The first term on the right is known to be negligible in comparison with the second term. This second term is the impulse $\bar{I}'_\tau$ that the particle receives during the time τ The average of this impulse is found next. From (A. 17) and (A.27) we obtain

$$I_\tau = \int_0^\tau \bar{\nabla}(\bar{d}'\cdot\bar{E})\,dt' = \sum_{\lambda'_1=1}^{2}\sum_{\lambda'_2=1}^{2}\int d^3k'_1\int d^3k'_2 f_1 h_2 \hat{\epsilon}_1\cdot\hat{\epsilon}_2 \frac{\tau}{2} k'_2 \times$$

$$\left\{\frac{Sin\ (\frac{\omega'_1+\omega'_2}{2}\tau)}{(\frac{w'_1+w'_2}{2})} Sin(\frac{w'_1+w'_2}{2}\tau-\theta'_1-\theta'_2-\alpha'_1) - \frac{Sin\ (\frac{\omega'_1-\omega'_2}{2}\tau)}{(\frac{\omega'_1-\omega'_2}{2})} Sin\ (\frac{\omega'_1-\omega'_2}{2}\tau-\theta'_1-\theta'_2-\alpha'_1)\right\} \quad \text{(A. 28)}$$

The subindices 1 and 2 denote the functions of $\{k'_1, \lambda'_1\}$ and $\{k'_2, \lambda'_2\}$ respectively. The average value of $\bar{I}'_\tau$ can be obtained from

$$\langle Cos\,\theta'(\bar{k}',\lambda')\rangle = \langle Sin\,\theta'(\bar{k}',\lambda')\rangle = 0 \quad \text{(A. 29)}$$

and because of (A. 14) and (A. 15) we can write that

$$\langle I'_\tau\rangle = 0\int d^3k\left(\frac{3c^3}{2\omega'^3}\right)h'^2[\bar{k}'] Sin^2(\alpha(\omega'))\,\bar{k}' \quad \text{(A. 30)}$$

where the summation over the λ subindex has already been carried out.

For a particle moving at relativistic speeds through a cavity that contains thermal radiation the expression for $h'[\bar{k}']$ of (A. 19) has to be used in (A. 30). However, if the motion is such that $v \ll c$, it is enought to introduce the approximate expression of (A.21)

$$\langle I'_\tau\rangle \cong -2\pi\tau\int\left(h^2(\omega')-\omega'\frac{\partial h^2(\omega')}{\partial\omega}\right) \langle Sin^2(\alpha(\omega'))\rangle \frac{\bar{\nu}}{c}\,d\omega' \quad \text{(A. 31)}$$

If we average over the terms of eq. (A. 1) when in thermal equilibrium we are left with

$$\bar{F} = \langle I_\tau\rangle = -R\,\bar{v}_t\,\tau \quad \text{(A. 32)}$$

since the averaging is over the field phase angle $\theta(\bar{k},\lambda)$ which is not present in the velocity of the particle. This gives, to first order in v/c and dropping the primed notation, the expression of eq. (A. 4), where a simple integration over ω has been carried out in which the highly peaked form of the expression for $Sin^2(\alpha(\omega))$ is taken into account in a manner previously explained. For the zero temperature case i.e., the case when eq. (A. 6) reduces to the ZPF case of (1) we obtain then

$$R = 0 \qquad (T=0,\ v/c \ll 1) \quad \text{(A. 33)}$$

Eq. (A. 33) is not quite satisfactory. It was derived from expressions whose validity was restricted to the $v/c \ll 1$ case. We expect physically at least that the result holds up to $v/c \cong 1$. This we obtain as follows. Replacing in (A. 30) the expression of the energy per mode of (A. 20) that corresponds to the ZPF we obtain

$$\langle I'_\tau\rangle = \tau\int\bar{k}d^3k'\left(\frac{3c^3}{2\omega'^3}\right)\left(\frac{\hbar\omega'}{2\pi^2}\right)Sin^2(\alpha(\omega')). \quad \text{(A. 34)}$$

Integratin next over the angles we obtain $\langle\bar{\Delta}'\rangle = 0$; This duely implies that the friction factor R is zero for the ZPF even when $v/c \cong 1$:

$$R = 0\ (T=0,\ 0 \leqslant v/c \leqslant 1). \quad \text{(A. 35)}$$

The case of monopolar particles was recently presented in the literature with a proof that (A. 35) also holds for monopolar particles[13]. If both monopolar and polarizable particles have been shown not to suffer frictional forces at all, even when $v \cong c$ and when in motion through the ZPF, it is not hard to guess that all electromagnetically interacting particles, and henceforth all particles, do not suffer frictional forces when moving through the electromagnetic ZPF[39].

APPENDIX B- REMARKS ON THE THERMODYNAMICS OF THE PROBABILITY FLOW

In this Appendix we further explore some thermodynamic relationships for the probability fluid analogy. This analogy permits us to pass, in the standard manner of the hydrodynamics of irrotational isentropic fluids[20,24], from eq. (23) to the Bernoulli equation (39) and from it to the Lagrangian density of (31). Consider a *fixed* and small amount of probability fluid with mass μ enclosed in a slightly variable or deformable small volume ν, we thus have

$$\mu = m\rho(\bar{r}, t)\nu \qquad \text{(B. 1)}$$

Let $e = \epsilon\rho\gamma = \epsilon\mu$ be the internal energy associated with the fluid element. For variations in internal energy the first law of thermodynamics gives

$$de = T\,dS - pd\nu = T\mu ds - pd\nu \qquad \text{(B. 2)}$$

We are considering an isentropic flow. Eqs. (10), (B. 1) and (B. 2) then give

$$de = \mu d\epsilon = \frac{\mu p}{m\rho^2}\,d\rho . \qquad \text{(B. 3)}$$

The internal energy per unit mass can thus be considered to be a function of (ρ, s) instead of (ν, s). We thus write

$$d\epsilon = \left(\frac{\partial\epsilon}{\delta\rho}\right)_s d\rho + \left(\frac{\partial\epsilon}{\partial s}\right)_\rho ds = \frac{p}{m\rho^2}\,d\rho \qquad \text{(B. 4)}$$

As the process is isentropic it is useful to use the enthalpy h of the fixed mass μ of fluid

$$h = e + p\nu \qquad \text{(B. 5)}$$

and thus

$$dh = T\mu ds + \nu\,dp \qquad \text{(B. 6)}$$

where the enthalpy per unit mass $\eta = h/\mu$ can be introduced. Thus, using (B. 5) and (B. 6), we have for the gradients of η, ϵ and p the relationships

$$\bar{\nabla}\eta = \bar{\nabla}\left(\epsilon + \frac{p}{m\rho}\right) = \frac{1}{m\rho}\,\bar{\nabla}p . \qquad \text{(B. 7)}$$

We can now go back to eq. (23), and realizing that for a conservative force field the force per unit mass is

$$\bar{f} = -\bar{\nabla}\,V(\bar{r}, t), \qquad \text{(B. 8)}$$

where $V(\bar{r}, t)$ is the potential per uni mass of eq. (13), we may write, using eqs. (23), (30), (B.7) and (B. 8),

$$\Delta\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}(\bar{\nabla}\Phi)^2 + \eta + V\right) = 0 . \qquad \text{(B. 9)}$$

This expression may readily be integrated giving the equation of Bernoulli for irrotational isentropic fluids

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}(\bar{\nabla}\Phi)^2 + \eta + V = 0 \qquad \text{(B.10)}$$

where the constant of integration has been absorbe in the potential. The equations of Bernoulli (B. 10) with that of continuity (3) can be obtained from the Lagrangian density of (31) as shown in Section 4. This is ultimately the justification for the Lagrangian.

BIBLIOGRAPHIC

1. G. CAVALLERI, preprints CISE (1981) (to be published).
2. E. MADELUNG, *Die Mathematischen Hilfsmittel des Physikers* (Springer, Leipzig 1922, 1925, 1935; Dover, New York, 1943) and Zeitschrift für Physik 40, 322 (1926).
3. D. BOHM, PHYS. Rev. 85, 166 and 180 (1952).
4. E. NELSON, PHYS. Rev. 150, 1079 (1966).
5. E. SANTOS, Nuovo Cimento 59B, 65 (1969).
6. L. DE LA PEÑA-AUERBACH and A. M. CETTO, J. MATH. Phys. 18, 1612 (1977).
7. M. JAMMER, *The Philosophy of Quantum Mechanics* Wiley, New York (1974).
8. T. W. MARSHALL, Proc. Cambridge Philos. Soc. 61, 537 (1965).
9. T.H. BOYER, Phys. Rev. 182, 1374 (1969).

10. T. H. BOYER, in *Foundations of Radiation Theory and Quantum Electrodynamics*, edited by A. O. Barut (Plenum, New York, 1980).

11. P. CLAVERIE and F. SOTO, J. MATH Phys (in press) and references therein.

12. T. H. BOYER, Phys. Rev. D. 11, 790 (1975).

13. A. RUEDA, Phys. Rev. A. 23, 2020 (1981).

14. I. FENYES, Z. PHYSIK 132, 81 (1952).

15. A. RUEDA, Nuovo Cimento 48A, 155 (1978).

16. A. RUEDA and A. LECOMPTE, Nuovo Cimento 52A, 264 (1979).

17. A. LECOMPTE and A. RUEDA, Revista Colombiana de Física (to be published).

18. A. RUEDA and G. CAVALLERI, Nuovo Cimento C 6, 239 (1983).

19. L. DE LA PEÑA-AUERBACH and A. M. CETTO, J. Math. Phys. 18, 1612 (1977); Found. Phys. 8, 191 (1978).

20. L. D. LANDAU and E. M. LIFSCHITZ *Fluid Mecnanics* (Pergamon, London, 1959).

21. A. EINSTEIN and L. HOPF, Ann. Phys. (Leipzig) 33, 1105 and 1096 (1910).

22. S. M. MOORE, Lettere Nuovo Cimento, 32, 91 (1981).

23. SEE, E. G., L. DE LA PEÑA-AUERBACH and A. M. CETTO, *Derivation of Quantum Mechanics from Stochastic Electrodynamics*, preprint IFUNAM 76-18 (1976).

24. A. L. FETTER and J. D. WALECKA, *Theoretical Mechanics of Particles and Continua* (McGraw-Hill, New York, 1980).

25. S. M. MOORE, FOUND, Phys. 9, 237 (1979); J. MATH. Phys. 21, 2102 (1980).

26. H. GRABERT, P. HANNGI and P. TALKNER, Phys. Rev. A. 19, 2440 (1979).

27. This name has sometimes been used for denoting the Abraham-Lorentz equation with a random or Stochastic driving force which usually refers to the electric component of the ZPF.

28. G. CAVALLERI, Phys. Rev. D 23, 363 (1981).

29. P. CLAVERIE and S. DINER, Int. J. Quantum Chemistry Vol. XII 41 (1977).

30. P. CLAVERIE in *Proceedings Einstein Centennial Simposium on Fundamental Physics* (Universidad de los Andes, Bogotá, 1981).

31. G. CAVALLERI, personal communication.

32. T. H. BOYER, Phys. Rev. 186, 1304 (1969).

33. A. RUEDA and J. A. TAUBER, Uniandes preprint (to be published).

34. See, e.g., T. S. KUHN, *Black-Body Theory and the Quantum Discontinuity* (Clarendon Press, Oxford, 1978).

35. The thermodynamics of thermal radiation fields has been examined in A. RUEDA, J. CHEM. Phys. 58, 3230 (1973); J. Opt. Soc. of Am. 63, 573 (1973); Phys. Rev. A. 8, 2099 (1973) and Found. Phys. 4, 215 (1974).

36. A. RUEDA - *Explicit enunciation of hypothesis in a fluid model for the one particle Schrödinger's equation in the context of Stochastic Electrodynamics in Stochastic Processes in Physics and Related Fields*, B. Gómez, S. M. Moore, A. M. Rodríguez-Vargas and A. Rueda, editors (World Scientific Publishers, Singapore, 1983).

37. A. RUEDA, Uniandes Preprint 56-11-83 (Nov. 1983) to be published; Phys. Rev. A 30, 2221 (1984).

38. G. CAVALLERI (1983), preprints to be published in Lettere Nuovo Cimento and in Nuovo Cimento A.

    Publishing in a very modified form and introducing Zitterbewegung as an additional hypothesis to those of ordinary SED in Lett. Nuovo Cimento issue of 16 July (1986).

39. A proff that this is true at all speeds for anharmonically bound polarizable particles is given in C. Díaz-Salamanca and A. Rueda, Phys. Rev D 29, 648 (1984).