A Winsorized Adaptive Rank Test for Location when Sampling from Asymmetric Distributions

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ABSTRACT

We propose a winsorized adaptive rank test for the location alternative for samples from asymmetric distributions coming from the Generalized Lambda Family. We give analytic expressions for the exact and asymptotic distributions and for the first two moments of the test statistic. By means of a Monte Carlo study, we show that for various selections of the winsorization parameter, our test is more powerful than the sign test, and than the original test, from which the proposed test is adapted.

Keywords: Location Tests, Winsorized Rank Tests, Power of Rank Tests.
Pruebas de Localización, Pruebas de Rangos Winsorizadas, Potencia de Pruebas de Rangos.

2000 Mathematics Subject Classification: 62G10, 62G10, 62G20, 62G35.

Una prueba de rangos adaptativa winsorizada para localización en muestras de distribuciones asimétricas

RESUMEN

Se propone una prueba de rangos adaptativa winsorizada para la alternativa de localización en distribuciones asimétricas que provienen de la Familia Lambda Generalizada. Se encuentran expresiones analíticas para la distribución exacta y asintótica, y los dos primeros momentos de la estadística de prueba. Por medio de un estudio de Monte Carlo, se muestra que para varias selecciones de parámetros de winsorización la prueba propuesta es más potente que la prueba del signo y que la prueba original de donde fue adaptada.

Palabras claves: Pruebas de Localización, Pruebas de Rangos Winsorizadas, Potencia de Pruebas de Rangos.

1

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1 Introduction

Let $X_1, \dots, X_N$, be a random sample from a continuous distribution $F(x-\theta)$, such that $F(0) = 1/2$ uniquely. Without loss of generality, consider the test problem:

$$H_0 : \theta = 0 \quad vs. \quad H_1 : \theta > 0,$$

or versus the alternatives $\theta < 0$ or $\theta \neq 0$. Under such general conditions on F, the sign test is a locally most powerful test for H_0 against H_1 when the sample distribution is double exponential (Hettmansperger 1984, page 9-12)) . When the symmetry of F around zero is justifiable, the Wilcoxon signed rank test is preferred, especially when the sampled distribution is logistic (Hajek 1999, page 119). Moreover, more efficient tests can be obtained by including information about the tail weight of the sampled distribution. This class of tests is called winsorized signed rank tests, and they are preferred instead of the Wilcoxon test, choosing a small winsorization parameter when the sampled distribution is close to the normal distribution, and a larger winsorization parameter when it is closer to the double exponential (Hettmansperger 1984, page 92-93). Baklizi 2005 proposed the use of the Wilcoxon scores modified by an exponent which depends on the asymmetry level of the sampled distribution to build a test for location under asymmetry, and showed by means of a simulation study that his test was more powerful than the sign test, than the Wilcoxon test, than the Lemmer 1987 and Lemmer 1993 tests, and than a bootstrap procedure test for samples coming from eight cases of the lognormal distribution.

We mix Baklizi's modification of the Wilcoxon scores with Tukey's winsorization technique to produce a new winsorized adaptive rank test, which becomes more powerful than the Baklizi test for samples coming from distributions with moderate levels of asymmetry obtained from the Generalized Lambda Distribution (GLD).

2 The Proposed Test Statistic and some Properties.

Let $|X|_{(1)} \leq \cdots \leq |X|_{(N)}$ be the sequence of ordered absolute values of the sample; define R_i, the rank of $|X_i|$, by $|X_i| = |X|_{(R_i)}$. Let $s(X_i)$ be the indicator variables:

$$s(X_i) = \begin{cases} 1 & \text{if } X_i > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

A general scores statistic is defined by (Hettmansperger 1984):

$$\bar{V} = \frac{1}{N} \sum_{i=1}^{N} \phi\left(\frac{R_i}{N+1}\right) s(X_i),$$

where $\phi(u)$, $0 < u < 1$, is a nonnegative and nondecreasing function such that $0 < \int_0^1 \phi^2(u)\, du < \infty$.

Some known special cases of $\bar{V}$ are: the sign test ($\bar{S}$-Test) statistic with $\phi(u) = 1$, the Wilcoxon signed rank test ($\bar{W}$-test) statistic with $\phi(u) = u$, the winsorized signed rank test ($\overline{WW}$-test) statistic with $\phi(u) = \min\{u, 1-\gamma\}$, $0 < \gamma < 1$, where γ is the proportion of winsorized observations.

A fourth special case of $\bar{V}$ is the Baklizi test ($\bar{B}$-test) statistic which uses the following conditional (on p) score function, $\phi(u) = u^p$, where p is the p-value of a test for the hypothesis of symmetry on F, $F(x) = 1 - F(-x)$, for all x against asymmetric alternatives, proposed in Randles, Fligner, Policello and Wolfe 1980, and is an indicator of the asymmetry of F. Note that when p approaches zero, F shows evidence of asymmetry. Explicitly, the Baklizi test statistic can be written as follows:

$$\bar{B} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{R_i}{N+1}\right)^p s(X_i).$$

The fifth special case of $\bar{V}$ is the proposed test ($\overline{BW}$-test) statistic, which uses the conditional (on p) score function:

$$\phi(u) = \min\{u^{g(p)}, 1-\gamma\}, \tag{2.2}$$

where $0 < \gamma < 1$ corresponds to the proportion of winsorized observations and $g(p)$ will be defined in what follows. The proposed test statistics are:

$$\overline{BW}_t = \frac{1}{N}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} s(X_i),\ t = 1,2, \tag{2.3}$$

where $\overline{BW}_1$ will be used for $g(p) = p$ and $\overline{BW}_2$ for $g(p) = \sqrt{p}$. When a property is common for the two statistics we will write $\overline{BW}$ without subscript.

Using Baklizi 2005 notation, let P be the random variable denoting the p-value of the Randles test for symmetry with probability density function $f(p)$. The proposed $\overline{BW}$-test rejects H_0 in favor of H_1 when $\overline{BW} \geq k$, where k is determined such that $P(\overline{BW} \geq k \mid p) = \alpha$. The overall size of the test is α because for a fixed γ (Baklizi, 2005):

$$P(\overline{BW} \geq k) = \int_0^1 P(\overline{BW} \geq k \mid p) f(p)dp = \alpha.$$

The exact conditional distribution of $\overline{BW}$ under H_0 can be obtained by enumeration as follows: Let Z be a $2^N \times N$ matrix containing all posible configurations of ones and zeros assignable to the sample values according to (2.1), obtained by the cartesian product $\{0,1\}^N$, such that each row corresponds to a different configuration. The distribution of the vector $(S(X_1), \ldots, S(X_N))$ is uniform on Z under H_0, because $X_1, \ldots, X_N$ are independent random variables from a continuous distribution with median zero. For $\phi\left(\frac{R_i}{N+1}\right)$ as in (2.2), let

$$R' = \left(\phi\left(\frac{1}{N+1}\right), \ldots, \phi\left(\frac{N}{N+1}\right)\right),$$

be a vector of scores, and denote by $z = (z_1, \ldots, z_N)$ a row vector representing a row of Z. The values of $\overline{BW}$ can be obtained as function of z by $\overline{BW}(z) = zR$, and so the distribution of $\overline{BW}$ can be calculated as:

$$P(\overline{BW}(z) \leq m \mid p) = \frac{\sharp(z \in Z : \overline{BW}(z) \leq m)}{2^N}.$$

The proposed test statistic has the following properties:

a) $\phi(u) = \min\{u^{g(p)}, 1-\gamma\} \leq 1-\gamma$; therefore,

$$\int_0^1 \phi(u)\,du < 1-\gamma, \quad \text{and} \quad 0 < \int_0^1 \phi^2(u)\,du < 1.$$

b) From the theory for linear rank statistics, the conditional mean and variance, exact and asymptotic, of $\overline{BW}$ for a given p under H_0 are (Hettmansperger 1984, page 88):

$$\begin{aligned} E\,[\overline{BW} \mid p\,] &= \frac{1}{2N}\sum_{i=1}^{N} \min\left\{ \left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma \right\} \\ &\xrightarrow{N\to\infty} \quad \frac{(1-\gamma)}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right], \end{aligned}$$

$$\begin{aligned} N\,Var\,[\overline{BW} \mid p\,] &= \frac{1}{4N}\sum_{i=1}^{N} \min^2\left\{ \left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma \right\} \\ &\xrightarrow{N\to\infty} \quad \frac{(1-\gamma)^2}{4}\left[1 - \frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]. \end{aligned}$$

(Details in Appendix A.1.)

c) It also holds that

$$\frac{\sqrt{N}\left(\overline{BW} - \frac{(1-\gamma)}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]\right)}{\sqrt{\frac{(1-\gamma)^2}{4}\left[1 - \frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]}},$$

converges to a standard normal distribution.

3 Monte Carlo Study

To study the empirical power of the proposed test, we selected four cases of the GLD with a moderate level of asymmetry, and to calibrate the size of the compared tests we selected the normal distribution approximated by the GLD. The parameters of these distributions are showed in Table 1 and the corresponding densities in Figures 1, 2, 3, 4, 5 and 6.

We have compared the $\bar{S}$-test, the $\bar{B}$-test, the $\overline{BW_1}$-test and $\overline{BW_2}$-test, and we have included the $\bar{W}$-test to calibrate the compared tests under the assumption of symmetry of the sampled distribution (case 1). In all cases, we used 0.05 as the significance level. The critical values for all compared tests were obtained from the normal distribution. For the simulation study, we adapted an algorithm to R code from Corzo and Babativa 2013, described as follows: generate a uniform random number u and calculate

$$x_i^* = \lambda_1 + \left(u_i^{\lambda_3} - (1-u_i)^{\lambda_4}\right)/\lambda_2, \quad i = 1, \ldots, n.$$

To center the simulated observations calculate, the median of the GLD as

$$\theta = \lambda_1 + \frac{0.5^{\lambda_3} - 0.5^{\lambda_4}}{\lambda_2},$$

and then center the data by $x_i = x_i^* - \theta$ so that $x_1, \ldots, x_n$ has zero median.

To calculate the empirical power of the compared tests, 1000 samples of size 30 were selected from each of the GLD cases, and we used the following values of the winsorization parameter: $\gamma =$ 0.1, 0.2, 0.4, 0.6, 0.8. The alternative hypothesis was simulated for values from $\theta = 0$ up to $\theta = 1$ for the cases two to five, and up to $\theta =$1.2 for the calibration case one, with steps of size 0.2. The values of p were calculated as in Baklizi 2005, from the modified test of Corzo and Babativa 2013.

Table 2 contains the empirical powers of the thirteen compared tests. In the calibration case 1, all tests tend to be conservative, excluding $\overline{BW}_2$(0.8), and they reach the maximum power for $\theta = 1$ or 1.2. Furthermore all $\overline{BW}$-tests show greater empirical powers than the $\bar{S}$-test. Moreover, the $\overline{BW}_2$(0.1) and $\overline{BW}_2$(0.2) show empirical powers greater than those of the $\bar{B}$-test.

In cases 2 to 6, the twelve compared tests tend to show an empirical size slightly greater than the nominal size 0.05. However, the empirical sizes of $\overline{BW}_1$(0.4) in case 2, $\overline{BW}_1$(0.8) in case 3, and $\overline{BW}_1$(0.6) in cases 4 and 5 are nearer to the nominal size 0.05 than the empirical size of the $\bar{B}$-test. Furthermore, for case 6, the empirical size of the $\overline{BW}_1$-tests, for any of the five values of γ, are lower or equal to the empirical size of the $\bar{B}$-test.

Moreover, the empirical sizes of the tests $\overline{BW}_2$(0.1) and $\overline{BW}_2$(0.2) in case 2, are slightly lower than that of the $\bar{B}$-test, but the empirical power of the $\overline{BW}_2$-test is greater than the empirical power of the $\bar{B}$-test in both cases. In case 3, the $\overline{BW}_1$(0.8)-test shows an empirical size nearer to the nominal size than the $\bar{B}$-test and its empirical power is very near to that of the $\bar{B}$-test. In case 4, the $\overline{BW}_2$(0.8) and $\overline{BW}_1$(0.6) tests have considerably lower empirical size than the $\bar{B}$-test. In case 5, the empirical size of the $\overline{BW}_2$-tests are lower than the empirical size of the $\bar{B}$-test; furthermore, the empirical size of the $\overline{BW}_1$(0.6)-test is nearer to the nominal size than that of the $\bar{B}$, although his empirical power is slightly lower than that of the $\bar{B}$. Finally in case 6, the tests $\overline{BW}_1$(0.4), $\overline{BW}_1$(0.6), $\overline{BW}_2$(0.6) and $\overline{BW}_2$(0.8) reach the exact nominal size, and their powers are almost equal to the empirical powers of the $\bar{B}$-test.

4 Conclusions and Discussion

In all studied cases there is at least one $\overline{BW}$-test with empirical size nearer the nominal size, with greater empirical power than the $\overline{S}$ test and almost as powerful as the $\bar{B}$ test.

In the calibration case, all proposed tests are well behaved in terms of their empirical powers and sizes, to the point that their empirical powers are greater that of the $\bar{S}$-test and are very near to the empirical powers of the $\bar{W}$-test. Additionally, the $\overline{BW}_2(0.1)$ and $\overline{BW}_2(0.2)$-tests reach better empirical powers than the $\bar{B}$-test.

We recommend the use of the $\overline{BW_1}(0.4)$ and $\overline{BW_1}(0.8)$-tests in cases 2 and 3 respectively. For cases 4, 5 and 6, the suggested tests are the $\overline{BW}_2(0.6)$.

To discussion, note that, with exception of the $\overline{S}$ test in cases 2 and 4 all other compared tests show tendency to be biased, but this tendency is lower for the $\overline{BW}_1$-tests. This can be due to the the functional form of $g(p)$, to the used test to select the value of p or to possible dependence between γ and p.

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A Proofs.

A.1 Conditional Mean and Variance of the Proposed Test Statistic under H_0.

Proof. Valid for $g(p) = p$ or $g(p) = \sqrt{p}$.

$$
\begin{aligned}
E\left[\overline{BW} \mid p\right] &= E\left[\frac{1}{N}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} s(X_i) \,\middle|\, p\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} E\left[s(X_i)\right] \\
&= \frac{1}{2}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} \frac{1}{N} \\
&\xrightarrow{N\to\infty} \quad \frac{1}{2}\int_0^1 \min\{u^{g(p)}, 1-\gamma\}\, du \\
&= \frac{(1-\gamma)}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right],
\end{aligned}
$$

$$
\begin{aligned}
N\,Var\left[\overline{BW} \mid p\right] &= N\,Var\left[\frac{1}{N}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} s(X_i) \,\middle|\, p\right] \\
&= \frac{N}{N^2}\sum_{i=1}^{N}\min^2\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} Var\left[s(X_i)\right] \\
&= \frac{1}{4}\sum_{i=1}^{N}\min^2\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} \frac{1}{N} \\
&\xrightarrow{N\to\infty} \quad \frac{1}{4}\int_0^1 \min^2\{u^{g(p)}, 1-\gamma\}\, du \\
&= \frac{(1-\gamma)^2}{4}\left[1 - \frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right].
\end{aligned}
$$

□

B Tables.

Table 1: Cases of the GLD

	λ_1	λ_2	λ_3	λ_4	α_3	α_4
Case 1	0	0.197454	0.134915	0.134915	0	3
Case 2	-0.116734	-0.351663	-0.13	-0.16	0.8	11.4
Case 3	0	-1	-0.1	-0.18	2.0	21.2
Case 4	3.586508	0.04306	0.025213	0.094029	0.9	4.2
Case 5	0	-1	-0.0075	-0.03	1.5	7.5
Case 6	0	1	0.00007	0.1	1.5	5.8

Table 2: Empirical Powers and Sizes of the Compared Tests

$\alpha = 0.05$					$\overline{BW}_1(\gamma)$					$\overline{BW}_2(\gamma)$				
Sampled distribution	θ	$\bar{W}$	$\bar{S}$	$\bar{B}$	$\overline{BW}_1(0.1)$	$\overline{BW}_1(0.2)$	$\overline{BW}_1(0.4)$	$\overline{BW}_1(0.6)$	$\overline{BW}_1(0.8)$	$\overline{BW}_2(0.1)$	$\overline{BW}_2(0.2)$	$\overline{BW}_2(0.4)$	$\overline{BW}_2(0.6)$	$\overline{BW}_2(0.8)$
Case 1	0	0.031	0.044	0.031	0.034	0.034	0.041	0.047	0.046	0.034	0.037	0.041	0.043	0.051
	0.2	0.269	0.217	0.251	0.247	0.250	0.249	0.246	0.237	0.257	0.253	0.261	0.259	0.247
	0.4	0.667	0.540	0.624	0.623	0.621	0.599	0.576	0.547	0.644	0.637	0.627	0.613	0.570
	0.6	0.927	0.833	0.890	0.886	0.884	0.870	0.857	0.843	0.911	0.903	0.884	0.874	0.854
	0.8	0.991	0.960	0.980	0.980	0.980	0.974	0.971	0.964	0.990	0.986	0.980	0.977	0.967
	1	1	0.997	1	1	1	0.999	0.999	0.997	1	1	1	1	0.997
	1.2	1	1	1	1	1	1	1	1	1	1	1	1	1
Case 2	0		0.061	0.060	0.061	0.060	0.056	0.059	0.061	0.059	0.059	0.063	0.061	0.061
	0.2		0.324	0.361	0.364	0.370	0.367	0.350	0.339	0.381	0.386	0.394	0.374	0.343
	0.4		0.756	0.816	0.807	0.800	0.783	0.773	0.763	0.840	0.841	0.829	0.804	0.773
	0.6		0.974	0.983	0.983	0.981	0.984	0.983	0.977	0.991	0.993	0.990	0.984	0.980
	0.8		0.994	0.999	0.999	0.997	0.994	0.994	0.994	0.999	0.999	0.999	0.997	0.994
	1		1	1	1	1	1	1	1	1	1	1	1	1
Case 3	0		0.050	0.054	0.056	0.054	0.054	0.054	0.051	0.060	0.057	0.059	0.057	0.054
	0.2		0.373	0.414	0.411	0.406	0.400	0.383	0.376	0.439	0.429	0.424	0.406	0.379
	0.4		0.834	0.857	0.854	0.850	0.846	0.844	0.840	0.877	0.881	0.863	0.847	0.843
	0.6		0.993	0.996	0.996	0.996	0.996	0.994	0.993	0.996	0.996	0.996	0.996	0.993
	0.8		1	1	1	1	1	1	1	1	1	1	1	1
	1		1	1	1	1	1	1	1	1	1	1	1	1
Case 4	0		0.059	0.069	0.067	0.067	0.064	0.059	0.060	0.079	0.077	0.071	0.063	0.060
	0.2		0.250	0.280	0.273	0.267	0.259	0.256	0.253	0.324	0.304	0.281	0.261	0.256
	0.4		0.657	0.709	0.700	0.686	0.671	0.661	0.659	0.746	0.727	0.699	0.680	0.663
	0.6		0.943	0.957	0.954	0.951	0.946	0.943	0.943	0.963	0.961	0.954	0.949	0.943
	0.8		1	1	1	1	1	1	1	1	1	1	1	1
	1		1	1	1	1	1	1	1	1	1	1	1	1
Case 5	0		0.051	0.057	0.056	0.054	0.053	0.051	0.053	0.056	0.056	0.054	0.053	0.053
	0.2		0.294	0.323	0.313	0.310	0.303	0.300	0.294	0.346	0.337	0.319	0.309	0.299
	0.4		0.756	0.777	0.773	0.764	0.761	0.756	0.756	0.801	0.790	0.776	0.764	0.756
	0.6		0.977	0.980	0.979	0.979	0.977	0.977	0.977	0.983	0.981	0.981	0.977	.977
	0.8		1	1	1	1	1	1	1	1	1	1	1	1
	1		1	1	1	1	1	1	1	1	1	1	1	1
Case 6	0		0.049	0.051	0.051	0.051	0.050	0.050	0.049	0.061	0.056	0.051	0.050	0.050
	0.2		0.279	0.284	0.280	0.280	0.280	0.279	0.279	0.311	0.300	0.280	0.280	0.279
	0.4		0.760	0.763	0.761	0.760	0.760	0.760	0.760	0.771	0.766	0.763	0.760	0.760
	0.6		0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999
	0.8		1	1	1	1	1	1	1	1	1	1	1	1
	1		1	1	1	1	1	1	1	1	1	1	1	1

C Figures.

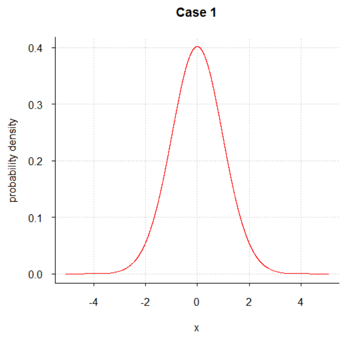

Figure 1: Density function for case 1 of the GLD

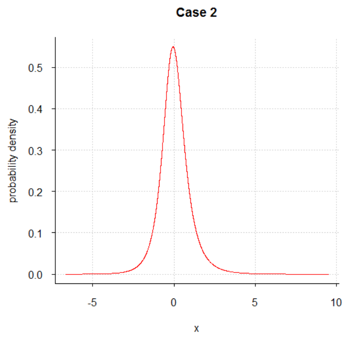

Figure 2: Density function for case 2 of the GLD

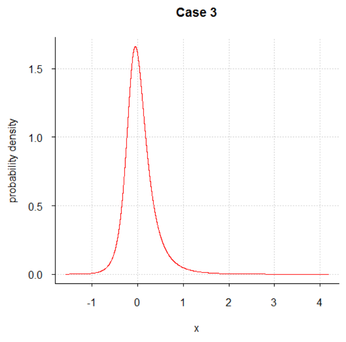

Figure 3: Density function for case 3 of the GLD

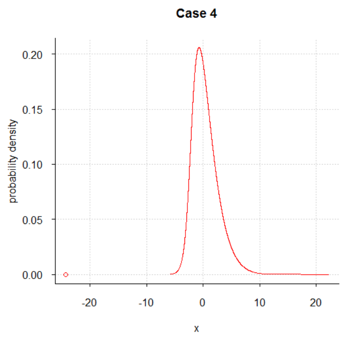

Figure 4: Density function for case 4 of the GLD

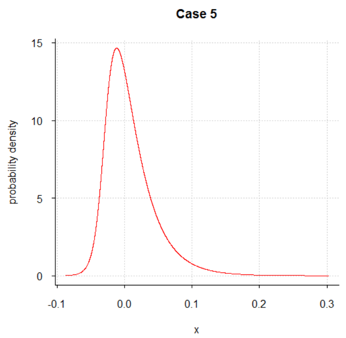

Figure 5: Density function for case 5 of the GLD

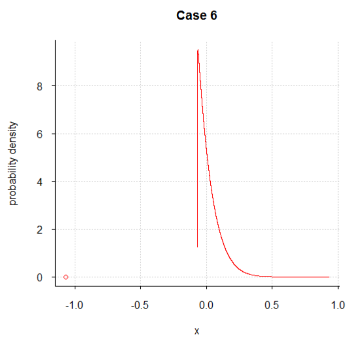

Figure 6: Density function for case 6 of the GLD