

SOME REMARKS ON THE LOCAL PATH-CONNECTEDNESS OF INFINITE POINT COMPACTIFICATIONS

por

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Resumen

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Se establecen algunos resultados sobre la conexión y la arco conexión locales de compactificaciones de espacios localmente compactos por adición de un número infinito de puntos, bajo restricciones sobre los puntos de acumulación de los conjuntos que se compactan.

Palabras claves: Conexión local, arco conexión local, σ -compacidad, paracompacidad.

Abstract

Local connectedness and local path-connectedness of infinite point compactifications of locally compact spaces are established under assumptions on the limit points of the compactifying sets.

Key words: Local connectedness, local path-connectedness, σ -compactness, paracompactness.

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1 Introduction

Compactifications of spaces are useful in diverse circumstances. Alexandroff's compactifications of locally compact Hausdorff spaces appear in many instances. More general compactifications (see [12], [13]) are usually needed when smoothness issues are involved. For example, the Alexandroff compactification of the cylinder

$$C = \{(x, y, z) \mid x^2 + y^2 = 1, |z| < 1\}$$

is a torus with a strangulation point (a sausage with identified end points) at which a conical neighborhood appears, and thus is not a topological manifold. However, a two point compactification of C is a sphere.

The transfer of local properties of a space to its compactifications is a frequent issue. For example, the transfer of local connectedness or of local path-connectedness is usually related to problems of analytic continuation and may be of importance.

In [4] it is shown that the local connectedness of a locally compact space is preserved by Alexandroff's compactification. This also holds for local path-connectedness of σ -compact or paracompact spaces. In [5], these results are extended to finite point compactifications under the additional assumption of local path-connectedness by closed neighborhoods.

The purpose of this paper is to extend the results in [4], [5] to infinite point compactifications. For infinite point compactifications matters are rather more delicate. For example, the space

$$X = \left\{ (x, y) \mid y = \sin\left(\frac{1}{x}\right), 0 < x \leq 1 \right\}$$

is locally compact, locally connected and locally path-connected for its topology of subspace of \mathbf{R}^2 , and $\tilde{X} = X \cup \{(0, y) \mid -1 \leq y \leq 1\}$, also with the subspace topology, is a compactification of X by addition of the infinite set $N = \{(0, y) \mid -1 \leq y \leq 1\}$. However, \tilde{X} is not locally connected at any point of N , and thus not locally path-connected at those points. As a matter of fact, \tilde{X} , though connected, is not path-connected. Therefore, conditions must be imposed on infinite point compactifications to preserve local connectedness or local path-connectedness. The conditions we give are appropriate for local path-connectedness. They may be relaxed if only local connectedness is sought. The main difficulty in dealing with local path-connectedness, as compared to local connectedness, lies in the fact that path-components may not be closed, and also in that the closure of a path-connected set may not be path-connected, as the above example shows. Nevertheless,

we will establish local path-connectedness of countably infinite compactifications, and even more general, under not very demanding constraints. As a matter of fact, we show that some of the assumptions in [5] can be relaxed. However, we think our results are only partial and not entirely satisfactory. It is difficult, for example, to construct significative examples to illustrate them. We suspect more natural conditions exist that can be expressed in terms of connectedness or path-connectedness properties of the compactifying sets. We have not succeeded in establishing these properties, though.

2 Preliminary notions and results

The terminology we follow is basically that of [3] or [6]. All topological spaces are supposed to be Hausdorff. If X is a topological space and $w \in X$, X is said to be *locally connected at w* if the connected neighborhoods of w are a fundamental system, i.e., if for any neighborhood U of w in X there is a connected neighborhood V of w such that $V \subseteq U$. If this holds for any point w in X , X is called *locally connected*. If Y is a subset of X and C is a connected subset of Y not properly contained in any other connected subset of Y , C is called a *connected component*, or, simply, a *component* of Y ; if $a \in C$, C is also called the *component of a in Y* and is denoted by $C_a(Y)$; if C is a component of Y and \bar{C} denotes the closure of C in X , it follows from the connectedness of $\bar{C} \cap Y$ ([3], p.124) that C is closed in Y . The space X is locally connected if and only if the components of open sets of X are open in X . If C is connected and A is an open and closed set in X , in which case we say that A is *clopen* in X , such that $C \cap A \neq \emptyset$, then $C \subseteq A$. Thus, any component of X is contained in any clopen set meeting it, and any component of a clopen subset of Y is a component of Y . If X is compact, any component of X is the intersection of the clopen sets containing it ([3], p.224). If X is a topological space and $Y \subseteq X$, a path in Y is a continuous map $\alpha : [0, 1] \rightarrow Y$, or, the same, a continuous map $\alpha : [0, 1] \rightarrow X$ such that $\alpha([0, 1]) \subseteq Y$. If $a, b \in Y$ and there is path α in Y such that $\alpha(0) = a$ and $\alpha(1) = b$, it is said that a and b can be joined by a path in Y . If any pair of points a, b in Y can be joined by a path in Y , Y is called a *path-connected subset* of X . Any path-connected set is also connected. The closure of a path-connected set may not be path-connected (see the example in Section 1). A path-connected subset of Y not properly contained in any other path-connected subset of Y is called a *path-component* of Y . Path-components of Y may not

be closed in Y (see example in Section 1). If C is a path-component of Y and $a \in C$, C is called the *path-component* of a in Y and is denoted by $C_a(Y)$. If C is a component of Y and B is a connected subset of Y such that $B \cap C \neq \emptyset$, then $B \subseteq C$. Hence $C_a(Y) \subseteq C_a(Y)$. If $w \in X$ and has a fundamental system of connected neighborhoods, X is said to be *locally path-connected* at w . If X is locally path-connected at every point, X is said to be *locally path-connected*. A space X is locally path-connected if and only if path-components of open sets are open. In a locally path-connected space, path-components are also closed. Thus, in a locally path-connected space, components and path-components coincide.

A *compactification* of a topological space X is a compact topological space \tilde{X} such that X is an open subspace of \tilde{X} and $\bar{X} = \tilde{X}$ (\bar{A} will always be the closure of A in \tilde{X}). We assume that X and \tilde{X} are Hausdorff spaces. The set $N := \tilde{X} - X$ is closed in \tilde{X} . Therefore, its set $D(N)$ of limit points is contained in N . Also $D^{(n)}(N) := D(D^{(n-1)}(N)) \subseteq D^{(n-1)}(N)$ and $D^{(n)}(N)$ is closed for each $n \geq 1$ (we also agree on $D^{(0)}(N) = N$). The set $D^{(n)}(N)$ is called the n^{th} -*derived set* of N . Points in $N - D(N)$ are said to be *isolated relative to N* . Clearly $w \in N - D(N)$ if and only if there is a neighborhood U of w in \tilde{X} such that $U \cap N = \{w\}$. For X to have a compactification in the above sense it is necessary that X be locally compact. Provided X is not compact, this condition is also sufficient for one point (Alexandroff) compactifications.

A compactification \tilde{X} of X is said to be *obtained from X by addition of the set $N = \tilde{X} - X$* . Observe that each point in N is in the closure of X . A space X is said to be *σ -compact (countable at infinity)* if it is locally compact and there a sequence (K_n) of compact subsets of X such that

$$X = \bigcup_{n=1}^{\infty} K_n.$$

Then (K_n) can taken such that $K_n \subseteq K_{n+1}^\circ$ (A° denotes the interior of A in X . See [3], [6]). A space X is called *paracompact* if it is locally compact and there are disjoint, σ -compact, open subsets X_α , $\alpha \in I$, of X such that

$$X = \bigcup_{\alpha \in I} X_\alpha.$$

A subset U of X is said to be *relatively σ -compact* if there a sequence (K_n) of compact subsets of X such that $K_n \subseteq K_{n+1}^\circ$ for all n and $U \subseteq \bigcup_{n=1}^{\infty} K_n$. We finally mention that if \mathcal{A} is a family of compact subset of X such

that $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$ for any finite $\mathcal{F} \subseteq \mathcal{A}$, then $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$ (Cantor's intersection theorem: [3], p.93).

3 Some basic results and observations

The following two elementary results on local connectedness will be needed.

Lemma 3.1. *If X is a locally connected compact space, then X has only a finite number of connected components; i.e., a compact space having an infinite number of connected components can not be locally connected.*

Proof. Components are pair-wise disjoint and, under the assumptions, they are also open in X . \square

Lemma 3.2. *If X has a locally connected compactification \tilde{X} , then X has only a finite number of compact connected components.*

If X has only a finite number of compact connected components and has a compactification \tilde{X} such that $N = \tilde{X} - X$ is finite, then \tilde{X} has only finitely many components.

Proof. Since compact connected components of X are connected components of \tilde{X} , the first assertion follows from Lemma 3.2. As to the second, just observe that the components of \tilde{X} meeting N are finite in number, and that a compact component of \tilde{X} not meeting N is a compact component of X . \square

Remark 3.1. We observe that a space X having infinitely many non-compact connected components may have locally connected compactifications (see Remark 3.3, next, or Example 5.1). Notice finally that the second assertion in Lemma 3.2 still holds if X has an infinite number of non-compact connected components.

Remark 3.2. A closed subset of a σ -compact space is obviously σ -compact. On the contrary, an open subset may not be σ -compact. For example, the Alexandroff compactification \tilde{X} of an infinite, non countable, discrete space X is σ -compact (it is compact). However, X is not σ -compact (recall that the compact subsets of X are the finite sets). Also, a closed subset of a paracompact space is paracompact, but the same does not hold for open subsets (for an example see [3], p.158, Exercise 12).

Remark 3.3. Let $X = \bigcup_{\alpha} X_{\alpha}$ be a paracompact space, where the X_{α} are disjoint, open and σ -compact. Then any component C of X is contained in some X_{α} (as X_{α} is clopen in X) and, being closed in X_{α} , is σ -compact. Thus, when X is locally connected, we may assume that each X_{α} is a component X . If I is infinite and only finitely many of the X_{α} are compact, a locally connected (locally path-connected) paracompact space may have locally connected (locally path-connected) compactifications. As matter of fact (see [3]), its Alexandroff compactification is locally connected (locally path-connected).

4 Local connectedness

Here we consider some results on local connectedness which will be important in Section 5. Although our approach in this and the next section is somewhat different, it is strongly motivated by ideas in [7] and [11]. In what follows, if W is a subset of the topological space

\tilde{X} , $Br(W) := \overline{W} \cap (\tilde{X} - W)$ will be the *boundary* of W , where \bar{A} stands for the closure of $A \subseteq \tilde{X}$ in \tilde{X} .

Lemma 4.1. *Let \tilde{X} be a connected compactification of the locally connected space X by addition of a set N . Let $w \in N$ be isolated relative to N and let U be an open neighborhood of w such that $U \cap N = \{w\}$. Let $\mathcal{E}(U) = U - C_w(U)$, where $C_w(U)$ is the connected component of w in U . Then $w \notin \mathcal{E}(U)$ and $\mathcal{E}(U)$ is a compact subset of X .*

Proof. We assume on the contrary that $w \in \overline{\mathcal{E}(U)}$ and let W be an open neighborhood of w such that $\overline{W} \subseteq U$. Any component of W is contained in a component of U . Let $\mathcal{E}'(W)$ be the union of those connected components of W which are contained in a component of $\mathcal{E}(U)$. Then $\mathcal{E}'(W) = \mathcal{E}(U) \cap W$. Therefore $w \in \overline{\mathcal{E}'(W)}$. Let C_w be the component of w in $\mathcal{E}'(W)$ and let A be a clopen subset of $\mathcal{E}'(W)$ such that $w \in A$. Since $A \cap \mathcal{E}'(W) \neq \emptyset$, A contains a component C of $\mathcal{E}'(W)$. Now, C is closed in $W - \{w\}$ and $w \notin C$. If $\overline{C} \cap Br(W) = \emptyset$, then $C = \overline{C}$ and thus C would be closed in \tilde{X} . Since C is open in $W - \{w\}$ and thus in W , then C would also be open in \tilde{X} . This is contradictory (as $\tilde{X} \neq C$). Thus, since $A \supseteq \overline{C}$ then $A \cap Br(W) \neq \emptyset$, and hence C_w , being the intersection of all such clopen A 's, will also meet $Br(W)$ ([3], p.224 and Cantor's intersection theorem). Since $w \notin Br(W)$, there is $x \in U$, $x \neq w$, such that $x \in C_w$. Now, since the component $C_x(U - \{w\})$ is open in \tilde{X} , and since

$x \in C_w \subseteq \overline{\mathcal{E}'(W)}$, then $C_x(U - \{w\}) \cap \mathcal{E}'(W) \neq \emptyset$. On the other hand, $C_w(U)$, the component of w in U , does not meet $\mathcal{E}(U)$. But C_w , being connected in U , is contained in $C_x(U)$, the component of x in U . Then $C_x(U) = C_w(U)$, and $C_x(U) \cap \mathcal{E}'(W) = \emptyset$. This is a contradiction, as $C_x(U - \{w\}) \subseteq C_x(U)$. Therefore $w \notin \mathcal{E}(U)$, and the lemma is proved. \square

Remark 4.1. Observe that in the above result it is important that \tilde{X} be connected. If \tilde{X} has infinitely many connected components, the conclusion may not hold. Let $X = \bigcup_{n=1}^{\infty} I_n$, where $I_n = \{\frac{1}{n}\} \times [0, \frac{1}{n}]$ with the topology of subspace of \mathbf{R}^2 , and let \tilde{X} be the Alexandroff compactification of X by addition of the point $w = (0, 0)$. This is just \tilde{X} as a subspace of \mathbf{R}^2 . Observe that $C_w(U) = \{w\}$ for any neighborhood U of w , but $\mathcal{E}(U)$ always contains a set $\bigcup_{n=m}^{\infty} I_n$, $m \geq 1$, and w is in the closure of such set. However, if \tilde{X} has only finitely many components, we may take U meeting only one of them, Y , and the argument above applies with Y in the place of \tilde{X} . Summing up, Lemma 3.1 holds if we assume that \tilde{X} may have finitely many components. We have:

Theorem 4.1. *If \tilde{X} is a connected compactification of the locally connected space X by a set N , and if $w \in N$ is isolated relative to N , then \tilde{X} is locally connected at w .*

Proof. Let U be an open neighborhood of w in \tilde{X} such that $U \cap N = \{w\}$. Since $w \notin \mathcal{E}(U)$ and the component $C_w(U)$ of w in U is $U - \mathcal{E}(U)$, there is a neighborhood V of w , $V \subseteq C_w(U)$. Hence, $C_w(U)$ is a connected neighborhood of U contained in U . \square

Remark 4.2. Observe that if U is open in \tilde{X} and $U \cap N = \{w\}$, then $C_w(U)$ is open in \tilde{X} . In fact, if $x \in C_w(U)$ and $x \neq w$, so that $x \in X$, the component $C_x(U - \{w\})$ of x in $U - \{w\}$ is open in \tilde{X} and contained in $C_w(U)$. Also observe that $\mathcal{E}(U)$ is closed in U and $\mathcal{E}(U) = \overline{\mathcal{E}(U)} \cap U$. Moreover,

$$C_w(U) = \bigcup_{x \in U - \mathcal{E}(U)} C_x(U - \{w\}) \cup \{w\},$$

w is in the closure of $C_x(U - \{w\})$ and $C_x(U) = C_w(U)$ for all $x \in U - \mathcal{E}(U)$.

Corollary 4.1. *Let \tilde{X} be a connected compactification of the locally connected space X by addition of N . Assume that N has only finitely many limit points in \tilde{X} .*

Then \tilde{X} is locally connected.

Proof. Any point in $N - D(N)$ is isolated relative to N , so that $Y = X \cup (N - D(N)) = \tilde{X} - D(N)$ is a locally connected open subspace of \tilde{X} , and \tilde{X} is a compactification of Y by addition of the finite set $D(N)$. Since $D(N)$, being finite, has no limit points, $\tilde{X} = Y \cup D(N)$ is locally connected at each point of $D(N)$. Thus, \tilde{X} is locally connected. \square

Corollary 4.2. *Let \tilde{X} be a connected compactification of the locally connected space X by addition of N . Also assume that $D^{(n)}(N) = \phi$ for some $n \geq 0$. Then \tilde{X} is locally connected.*

Proof. From above, the assertion is true for $n = 0, 1, 2$, and it follows by induction on n for $n \geq 3$, observing that \tilde{X} is a compactification of the locally connected space $Y = X \cup (N - D(N))$ by addition of $M = D(N)$, and $D^{(n-1)}(M) = \phi$. \square

Remark 4.3. If \tilde{X} is not connected but has instead finitely many components, and the remaining assumptions of Theorem 4.1 and its corollaries hold, each component of \tilde{X} will be locally connected. Hence, \tilde{X} itself will still be locally connected; i.e., the conclusions of Theorem 4.1 and its corollaries hold if X is locally connected and \tilde{X} is required to have only finitely many connected components. Because of Lemma 3.1 it is not to the point to let \tilde{X} to have infinitely many connected components.

5 Local path-connectedness

Now we come to the main results in this paper. Recall that if X is locally path-connected then any connected open subset of X is path-connected. In particular, components and path-components of X are open and coincide.

Lemma 5.1. *Let X be locally compact and locally path-connected, and let \tilde{X} be a compactification of X by addition of a set N . Assume that \tilde{X} has only finitely many connected components and let $w \in N - D(N)$. If w has a countable fundamental system of neighborhoods, then \tilde{X} is locally path-connected at w .*

Proof. The space \tilde{X} is locally connected at w (Theorem 4.1 and Remark 4.3). Let U be an open connected

neighborhood of w such that $\bar{U} \cap N = \{w\}$, and let (U_n) be a fundamental system a neighborhood of w . There is no loss of generality in assuming that the U_n are open and connected and that $U_{n+1} \subseteq U_n \subseteq U_0 = U$ for each $n \geq 0$. Let $a = a_0 \in U, a \neq w$. The path-component $C_a(U - \{w\})$ is clopen in $U - \{w\}$ and open in U . If $w \notin C_a(U - \{w\})$, it would also be closed in U , which is absurd. Then let $a_1 \in C_a(U - \{w\}) \cap U_1, a_2 \in C_{a_1}(U_1 - \{w\}) \cap U_2$, and continuing this way, let $a_{n+1} \in C_{a_n}(U_n - \{w\}) \cap U_{n+1}, n \geq 2$. Now choose (t_n) in $[0, 1)$ such that $t_n < t_{n+1}, t_0 = 0$ and $t_n \rightarrow 1$. Since $a_n, a_{n+1} \in C_{a_n}(U_n - \{w\})$, there is a path $\alpha_n : [t_n, t_{n+1}] \rightarrow U_n - \{w\}$ such that $\alpha_n(t_n) = a_n, \alpha_n(t_{n+1}) = a_{n+1}$. Let $\alpha : [0, 1] \rightarrow U$ be defined by $\alpha(t) = \alpha_n(t)$ if $t_n \leq t \leq t_{n+1}, \alpha(1) = w$. Clearly α is continuous in $[0, 1)$. It is also continuous at $t = 1$. In fact, $\alpha(t) \in U_m$ for $t \geq t_m$, and (U_n) is a fundamental system of neighborhoods of w . Hence α is a path of U , and thus U is path-connected. This proves the lemma. \square

Remark 5.1. It follows that for any open neighborhood of w such that $U \cap N = \{w\}$ we have $C_w(U) = C_x(U) = C_x(U)$ for all $x \in U - \mathcal{E}(U)$; and if U is connected, it is also path-connected.

Corollary 5.1. *Let X be locally path-connected and let \tilde{X} be a compactification of X by addition of a finite set N . Assume that X has only finitely many compact connected components. Also assume that each $w \in N$ has a countable fundamental system of neighborhoods. Then \tilde{X} is locally path-connected.*

Proof. Under the assumptions \tilde{X} has only finitely many components (Lemma 3.2), and each point of N is isolated relative to N . \square

Corollary 5.2. *Let X be locally compact and locally path-connected, and let \tilde{X} be a compactification of X by addition of a set N having only finitely many limit points. If \tilde{X} has only finitely many components and if any point in N has a countable fundamental system of neighborhoods, then \tilde{X} is locally path connected.*

Proof. The space \tilde{X} is the compactification of the locally path-connected subspace $Y = X \cup (N - D(N))$ of \tilde{X} by addition of $D(N)$, which is finite. \square

By induction it follows as before that:

Theorem 5.1. *If X is locally compact and locally path-connected, if \tilde{X} is a compactification of X by addition*

of a set N such that $D^{(n)}(N) = \phi$ for some $n \geq 0$, if \tilde{X} has only finitely many components, and if any point in N has a countable fundamental system of neighborhoods, then \tilde{X} is locally path-connected.

Corollary 5.3. *If X is locally compact and locally path-connected, if \tilde{X} is a compactification of X by addition of a set N such that $D^{(n)}(N) = \phi$ for some $n \geq 0$, if \tilde{X} has only finitely many components, and if for each $w \in N$ there is a neighborhood U such that $\bar{U} - \{w\}$ is relatively σ -compact in $\tilde{X} - \{w\}$, then \tilde{X} is locally path-connected.*

Proof. Since the space X is locally connected, so is the open subspace $Y = \tilde{X} - \{w\}$. Observe that \tilde{X} is the Alexandroff compactification of Y by addition of w and that $\bar{U} - \{w\} \subseteq \bigcup_{n=1}^{\infty} K_n$, where the K_n 's are compact subsets of Y such that $K_n \subseteq K_{n+1}^{\circ}$ for each $n \geq 1$. Let $\mathcal{E}_n(U) = K_n \cup \mathcal{E}(U - K_n)$. Then $\overline{\mathcal{E}_n(U)}$ is compact subset of Y . Let $U_n = U - \overline{\mathcal{E}_n(U)}$. Since any neighborhood V of w in U contains one of the form $U - K$ where K is a compact subset of $\bar{U} - \{w\}$, and since $K \subseteq K_n$ for some n , so that $V \supseteq U_n$, it follows that (U_n) is a fundamental system of path-connected neighborhoods of w in \tilde{X} . \square

Corollary 5.4. *If X is locally path-connected and σ -compact, and if \tilde{X} is a compactification of X by addition of a countable set N such that $D^{(n)}(N) = \phi$ for some $n \geq 0$, then, provided it has only finitely many components, \tilde{X} is locally path-connected.*

Proof. Let $w \in N$. We may assume $N - \{w\} = \{w_n | n \geq 1\}$, where $w_n \neq w_m$ for $m \neq n$. Let (K_n) be a sequence of compact subset of X such that $X = \bigcup_{n=1}^{\infty} K_n$ and let $K'_n = K_n \cup \{w_n\}$. Then (K'_n) is a sequence of compact subsets of $\tilde{X} - \{w\}$ such that $\tilde{X} - \{w\} = \bigcup_{n=1}^{\infty} K'_n$. Thus, any neighborhood U of w is such that $\bar{U} - \{w\}$ is locally σ -compact in $\tilde{X} - \{w\}$. \square

Remark 5.2. Corollary 5.1 was proved in [5] under the additional assumption that any point $x \in X$ had a fundamental system of closed path-connected neighborhoods. This assumption thus proves to be superfluous.

Corollary 5.5. *Assume X is paracompact and locally path-connected, and let $(X_\alpha)_{\alpha \in I}$ be the family of its distinct connected components. Let \tilde{X} be a connected compactification of X by addition of a set N and assume \tilde{X} has only finitely many compact components. Then \tilde{X}*

is locally path-connected at any point $w \in N - D(N)$.

Proof. In fact, there is an open connected neighborhood U of w with $\bar{U} \cap N = \{w\}$. For each $\alpha \in I$, let $U_\alpha = U \cap \overline{X_\alpha}$ and $U'_\alpha = C_w(U_\alpha)$. Observe that $U'_\alpha = \phi$ if $w \notin \overline{X_\alpha}$, i.e., if $w \notin U_\alpha$. On the other hand, $U_\alpha = U'_\alpha$ if $w \in U_\alpha$. In fact, $U_\alpha = (U \cap X_\alpha) \cup \{w\}$, and any component C of $U \cap X_\alpha$ such that $w \notin \bar{C}$, being clopen in U , is a component of U . Hence w is in the closure of any component of $U \cap X_\alpha$, and therefore U_α is connected. Since $\bar{U}_\alpha - \{w\} = \bar{U} \cap X_\alpha$ is relatively σ -compact, it follows that $U'_\alpha = U_\alpha$ is also path-connected (Remark 5.1). Finally, since $U = \bigcup_{\alpha \in I} U'_\alpha$, and the latter set is path-connected, the assertion follows.

Theorem 5.2. *Let X be a locally path-connected and paracompact space and let \tilde{X} be a compactification of X by addition of a finite set N . Also assume that X has only finitely many compact components. Then \tilde{X} is locally path-connected.*

Proof. Since \tilde{X} has only finitely many components (Lemma 3.2), and since $D(N) = \phi$, so that $N - D(N) = N$, \tilde{X} is locally path-connected. \square

Remark 5.3. Theorem 5.2 is Theorem 4.2 in [5], proved without the assumption that points in X have fundamental systems of closed path-connected neighborhoods.

Remark 5.4. Let $X = \bigcup_{\alpha \in I} X_\alpha$ be a locally connected paracompact space, the X_α being the components of X . Let \tilde{X} be a compactification of X by addition of an infinite set N , so that $D(N) \neq \phi$, and assume \tilde{X} has only a finite number of components. The set $D(N)$ of limit points of N is closed in \tilde{X} . Hence, $Y := \tilde{X} - D(N) = X \cup (N - D(N))$ is an open subset of \tilde{X} , and therefore a locally compact subspace of \tilde{X} . Furthermore, Y is locally connected, as follows from Theorem 4.1. Thus, if $w \in N - D(N)$, the component $C_w(Y)$ of w in Y is of the form $C_w(Y) = N_w \cup \bigcup_{\alpha \in I_w} X_\alpha$, where $N_w \subseteq N - D(N)$, $I_w \subseteq I$ and each point $x \in N_w$ is in the closure of $\bigcup_{\alpha \in I_w} X_\alpha$ (if not, there would be an open neighborhood W of x such that $\bar{W} \cap C_w(Y) = W \cap C_w(Y) = \{x\}$, and $\{x\}$ would be clopen in $C_w(Y)$). As a matter of fact, x is in the closure of some X_α , $\alpha \in I_w$. To see this, let U be an open connected neighborhood of x such that $\bar{U} \cap N = \{x\}$. Since x is in the closure of $\bigcup_{\alpha \in I_w} X_\alpha$ then $U \cap X_\alpha \neq \phi$ for some $\alpha \in I_w$, and some component C_α of $U - \{x\}$ has to be contained in X_α . If $x \notin \bar{C}_\alpha$, then C_α would be clopen in U (it is

clopen in $U - \{x\}$), which is absurd since U is connected. Hence $x \in \overline{C_\alpha}$, and therefore $x \in \overline{X_\alpha}$. Also observe that since Y is locally connected, $C_w(Y)$ is open in Y , and hence in \tilde{X} , for all $w \in N - D(N)$. Now assume that I_w and N_w are countable. Then $\bigcup_{\alpha \in I_w} X_\alpha$ is σ -compact and therefore the union of a sequence (K_n) of compact subset of X . Let $N_w = \{x_n | n \geq 1\}$. Then $C_w(Y)$ is the union of the compact subsets $K_n \cup \{x_n\}$ of \tilde{X} and is also σ -compact. Thus, if I_w and N_w are countable for all $w \in N - D(N)$ then all the components of Y are σ -compact. Thus, $Y = X - D(N)$ is paracompact and locally path-connected (Corollary 5.5). Therefore, the following result holds.

Theorem 5.3. *Let X be paracompact and locally path-connected, and let \tilde{X} be a compactification of X by addition of an infinite set N having only finitely many limit points in \tilde{X} . Also assume that \tilde{X} has only finitely many components and that for each $w \in N - D(N)$ the sets I_w and N_w in Remark 5.4 are countable. Then, \tilde{X} is locally path-connected.*

A situation in which Theorem 5.3 can be applied is the following.

Example 5.1. Let $I = [-\pi, \pi]$ and for each $\theta \in I$ let L_θ be the subset of \mathbf{C} (the complex numbers) given by $L_\theta := \{re^{i\theta} | 0 < r < 1\}$, $re^{i\theta} = r(\cos \theta + i \sin \theta)$. Endow L_θ with the topology of subspace of \mathbf{C} , and let $X := \bigcup_{\theta \in I} L_\theta$ be given the sum topology ($A \subseteq X$ is open in X if and only if $A \cap L_\theta$ is open in L_θ for each θ . See [3], p.33). Then X is paracompact and not σ -compact. Let $N = \{e^{i\theta} | \theta \in I\} \cup \{0\}$. Also let $\tilde{L}_\theta := L_\theta \cup \{e^{i\theta}\}$ with its subspace topology and $Y := \bigcup_{\theta \in I} \tilde{L}_\theta$ with the sum topology. Finally let \tilde{X} be the Alexandroff compactification of Y by addition of $\{0\}$. Then \tilde{X} is a compactification of X by addition of N . Since X is paracompact and locally path-connected, and since N has 0 as its only limit point, then \tilde{X} is locally path-connected. Observe however that since Y is paracompact, this assertion also follows from Theorem 5.2. We are missing at the present time a simple more typical example of Theorem 5.3. Also observe that \tilde{X} is the unit disc in \mathbf{C} , but its topology is not that of a subspace of \mathbf{C} .

Remark 5.5. Theorem 5.3 most naturally applies when N is infinite countable, $D(N)$ is finite, and each point $w \in N - D(N)$ is in the closure of at most countably many of the X_α . This is trivially the case of the compactification $\tilde{X} = [0, 1]$ of $X = (0, 1) - M$ (with

their topologies of subspaces of \mathbf{R}), where $M = \{\frac{1}{n} | n = 1, 2, \dots\} \cup \{\frac{n}{n+1} | n = 1, 2, \dots\}$. In fact, \tilde{X} is obtained from X by addition of $N = M \cup \{0, 1\}$. Observe that in this case 0, 1 are not in the closure of any component of X , while all the other points in N are in the closure of just two such components. Of course, \tilde{X} is locally path-connected.

Theorem 5.4. *If X is a locally path-connected paracompact space, if \tilde{X} is a locally connected compactification of X by addition of a countable set N , and if each point $w \in N - D(N)$ is in the closure of at most countably many components of X , then $\tilde{X} - D(N)$ is paracompact and locally path-connected. Furthermore, if N has only finitely many limit points in \tilde{X} , then \tilde{X} is locally path-connected.*

Proof. We only need to prove that the sets I_w , $w \in N - D(N)$, in Remark 5.4, are countable. But this is trivial, as, with the notations in Remark 5.4, any X_α , $\alpha \in I_w$, must have some $x \in N_w$ in its closure in \tilde{X} (if this were not the case, X_α would be clopen in $C_w(Y)$), and N_w is obviously countable. \square

Remark 5.6. Under the assumptions of Theorem 5.4, if the number of components of X is not countable, some $w \in D(N)$ has to be in the closure of an uncountable number of X'_α s.

Theorem 5.4 can be further generalized.

Theorem 5.5. *Let X be locally path-connected and paracompact, and let \tilde{X} be a compactification of X by addition of a countable set N such that $D^{(n)}(N) = \emptyset$ for some $n \geq 1$. Also assume that each point in $N - D^{(n-1)}(N)$ is in the closure of at most countably many connected components and that \tilde{X} has finitely many connected components. Then, \tilde{X} is locally path-connected.*

Proof. For each $m \leq n - 1$ let

$$Y^{(m)} = X \cup \bigcup_{k=1}^m (D^{(k-1)}(N) - D^{(k)}(N)).$$

Since $D^{(k-1)}(N) - D^{(k)}(N) \subseteq D^{(0)}(N) - D^{(n-1)}(N) = N - D^{(n-1)}(N)$, it follows from Theorem 5.4 that $Y^{(1)}$ is paracompact, locally path-connected and, since any compact connected component of $Y^{(1)}$ is a connected component of \tilde{X} , $Y^{(1)}$ may have only finitely many com-

compact components. Assume the same holds for all $m <$

$n - 1$. Then

$$Y^{(n-1)} = Y^{(n-2)} \cup \left(D^{(n-2)}(N) - D^{(n-1)}(N) \right)$$

and we have to prove that any component of $Y^{(n-1)}$ is σ -compact. Let $w \in Y^{(n-1)}$. Then

$$C_w \left(Y^{(n-1)} \right) = N_w \cup \bigcup_{\alpha \in I_w} X_\alpha$$

where $N_w \subseteq D^{(n-2)}(N) - D^{(n-1)}(N) \subseteq N - D^{(n-1)}(N)$ is countable and each one of its points is in the closure of at most countably many of the X'_α 's. But, as in the discussion in Remark 5.4, any X_α , $\alpha \in I_w$, has a point $x \in N_w$ in its closure. Thus, I_w is countable and $C_w(Y^{(n-1)})$ is σ -compact. Hence, $Y^{(n-1)}$ is paracompact, locally path-connected with only finitely many compact components, and since $\tilde{X} = Y^{(n-1)} \cup D^{(n-1)}(N)$ and $D^{(n-1)}(N)$ is finite, it follows from Theorem 5.2 that \tilde{X} is path-connected. \square

Remark 5.7. The conditions we have given so far on N point in the direction of strong disconnectedness of this set. In fact, if $D^{(n)}(N) = \phi$, $w \in N$, and $C_w(N)$ is the connected component of w in N , also $D^{(n)}(C_w(N)) = \phi$, and we may assume that $D^{(n-1)}(C_w(N)) \neq \phi$. But, since $C_w(N)$ is connected, it has no isolated points. Thus $C_w(N) = D^{(1)}(C_w(N)) = D^{(2)}(C_w(N)) = \dots = D^{(n-1)}(C_w(N))$, and since this set is finite, it has to reduce to a point; i.e., N is *totally disconnected*. However, we have not succeeded in establishing local path-connectedness of \tilde{X} from total disconnectedness of N . We think the issue deserves further research.

Remark 5.8. Additional results on the local connectedness of compactifications can be found in [1] and [2]. The case of the Stone-C ech compactification of a completely regular (not necessarily locally compact) space poses a different kind of problem, as the space is not necessarily open in the compactification, but it is dealt with in [8] and [10]. It seems that little can be said about local path-connectedness in this case. The relationship between local connectedness and local path-connectedness within the frame of metric space is examined in detail in [9].

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