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Physical Sciences

Original article

Electrostatic Energy for a Circular Charged Configuration: Discrete vs. Continuum

Energía Electrostática para una Configuración de Carga Circular: Caso Discreto vs. Continuo

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Resumen

El concepto de energía potencial electrostática ha servido como punto de partida para un mayor entendimiento de varios modelos en teoría electromagnética. Usamos el concepto de energía potencial electrostática buscando una conexión entre varios de estos modelos, en particular, analizamos la contribución de la energía potencial electrostática para diferentes configuraciones de carga eléctrica con simetría circular. Estudiamos los modelos asociados a un conjunto de cargas discretas posicionadas en una figura poligonal, una distribución unidimensional de carga continua, y un toroide con carga continua, considerando las analogías que existen entre ellos. También se buscan los contrastes que existen entre las situaciones discretas y continuas de los modelos.

Palabras clave: Electromagnetismo; Energía potencial electrostática; Simetría circular.

Abstract

The concept of electrostatic potential energy has served as a starting point for a deeper understanding of several models in electromagnetic theory. We use this concept looking for a connection between some of those models, in particular, we analyze the contribution of electrostatic potential energy for different configurations of electric charge with circular symmetry. We study models corresponding to a discrete set of point charges arranged in a polygonal figure, a uni-dimensional continuously charged ring and a continuously charged torus, considering the existing analogies between them. In addition, we contrast the situation between the discrete and continuum models.

Keywords: Electromagnetism; Electrostatic potential energy; Circular symmetry.

Introduction

Interactions between elementary particles are one of the most fundamental features in physics to analyze the evolution and structure of several physical systems presented in nature (**Feynman, Leighton, & Sands,** 1963; **Griffiths,** 1999; **Kittel, Knight, & Ruderman,** 1973).

Besides the gravitational force, the electromagnetic interaction has been widely studied since ancient times. It is the sole source of forces between atomic nuclei and orbiting electrons, and the reason for the existence of electromechanical devices. For two stationary objects with an excess charge which are located in the same reference frame, they manifest attraction or repulsion. If they have opposite sign charge, the force between them is attractive, and if they have the same sign charge, the force is repulsive.

The fundamental law that postulates that equal charges repel and different charges attract each other, has served as an elementary law for the development or operation of other experiments in physics (**Greiner,** 1998; **Kittel, Knight, & Ruderman,** 1973). On the other

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hand, through the concept of work in mechanics, a consistent definition of energy, which is also fundamental in physics, has been established (**Wells**, 1967; **Whittaker**, 1917). Energy is defined as the work done by an applied force on a particle that leads to a change in its spatial configuration. In the context of electrical forces, this is known as electrostatic potential energy or simply potential energy (**Griffiths**, 1999). The concept of potential energy plays an important role in modern technology advances with implications in several branches of science. For example, it plays a significant role, if we are interested in processes occurring in molecular mechanics such as composition and charge interacting in solvents, ionized media, and on surfaces of molecular structures (**Besley**, 2023), or describing the nature of noncovalent and protein–protein interactions which are essential in chemical and biological processes (**Brinck** & **Borrfors**, 2022; **Maleki**, **Vasudev**, & **Rueda**, 2013).

In this article, we are interested in studying the potential energy stored in different geometrical charge configurations with circular symmetry in the electrostatic case. We consider classical charged particles whose interactions are equal to the sum of Coulomb forces only. Even though the static interaction force between charged particles originates from the electric field-induced interactions, here we do not derive results using an electric field scheme.

This work is organized as follows. In section *theoretical framework*, we introduce an elementary definition of electrostatic potential energy based on classical literature (**Feynman**, **Leighton**, & **Sands**, 1963; **Good**, 1999; **Griffiths**, 1999; **Wangsness**, 1997). In section *potential energy for a discrete arrangement of point charges*, we describe the energy stored in a set of point charges in electrostatic equilibrium in empty space, the one-dimensional electrostatic interaction of several point charges located at vertices of a polygon, and inscribed into a unitary imaginary circle interacting with another point charge located along the axis of the circle and out of the plane. The analysis of the electric potential energy of a uniformly charged ring interacting with a point charged particle located along the axis of the ring and out of the plane is studied in section *electrostatic energy for an a uniformly charged ring and a point charge*. In section *electrostatic energy for a uniformly charged torus and a point charge*, we study a similar charge configuration as in the aforementioned section but this time with a uniformly charged torus instead of a uniformly charged ring. All of the different configurations above have an axial symmetry, which allow us to explore possible similarities between them. Finally, we present our conclusions in the last section.

Theoretical framework

We consider the work done by an electrical force that interacts with a test charge to change its position from position a to position b. We know from classical mechanics (**Feynman**, **Leighton**, & **Sands**, 1963; **Kittel**, **Knight**, & **Ruderman**, 1973; **Wells**, 1967) that the work done by a force to move an object from position a to position b is

$$W = \int_a^b \mathbf{F} \bullet d\mathbf{l}. \tag{1}$$

If there is a charge interacting with an electric field, then the force expression is $F = qE$. Here q is the electrical charge, and E is the magnitude of the electrical field. It is well known that the electrostatic force is conservative, thus, the total work is path independent, and depends only on the initial and final position of the electric charge in the field, that means, no matter what the election path is, the total work depends only on the initial and final positions of the electric charge in the field, that is $W = q(V(b) - V(a))$, where $V(a)$ and $V(b)$ are known as the electric potentials at positions a and b, respectively (**Griffiths**, 1999; **Wangsness**, 1997). For an individual charge, we say that it is moved from a far position in which the electric potential is zero, to a final position r from the origin of a coordinate system. In this form, the expression for the electric work is $W = qV(r)$. We will use this

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last definition for all cases described later. If we study the case of two point charges, then the electrical work stored in the electric field is

$$W_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|}, \tag{2}$$

where ε_0 is electrical permittivity of vacuum, q_1, q_2 are point charges and $\vec{r}_1 = (x_1, y_1, z_1)$, $\vec{r}_2 = (x_2, y_2, z_2)$ are charge positions, respectively. We can say that electric charge number two has move to actual position in the presence of the field generated by electrical charge number one, mathematically, $W_1(\vec{r}_2) = q_2 V_1(\vec{r}_2)$ or vice versa, $W_2(\vec{r}_1) = q_1 V_2(\vec{r}_1)$. This last definition allow us to rewrite the expression for the potential energy for a system of N point charges at electrostatic equilibrium. Since the electric force is conservative, the potential energy for a system of N point charges at electrostatic equilibrium can be expressed as

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{i=2}^{N}\sum_{k=1}^{i-1}\frac{q_i q_k}{|\vec{r}_i - \vec{r}_k|}. \tag{3}$$

Another equivalent relation is

$$U = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{k=1 \\ k\neq i}}^{N}\frac{q_i q_k}{|\vec{r}_i - \vec{r}_k|}, \tag{4}$$

where the terms $i = k$ are omitted. If we rearrange terms we obtain

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i), \tag{5}$$

with

$$V(\vec{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{k=1 \\ k\neq i}}^{N}\frac{q_k}{|\vec{r}_i - \vec{r}_k|}, \tag{6}$$

this last expression corresponds to the electric potential due to the kth charge system in the ith charge position, respectively (**Greiner**, 1998; **Griffiths**, 1999; **Wangsness**, 1997).

Potential energy for a discrete arrangement of point charges

We consider the situation of a charge distribution consisting of several discrete point charges in a plane and located at the vertices of a polygon inscribed into an imaginary circle with radius a, and a single point charge out of the charge distribution plane and placed over the axis of the circle perpendicular to the plane. A similar approach is proposed in (**Antonov**, 2003).

We propose the potential energy of such charge distribution as the sum of three energetic terms that we call three dimensional energy u_{trid}, polygonal energy u_{polig}, and diametrical energy u_{diam}. The three dimensional energy refers to the sum of energy terms due to the interaction between the axial point charge at height h on the z axis, and the rest of the charges located in the plane of the circle in the xy plane. The polygonal energy is the sum of potential energy terms produced by the interaction of neighboring charges located on the circle. The diametrical energy is produced by the interaction of diametrically opposite charges, thus, the total potential energy is

$$U_{Total}(n) = u_{trid}(n) + u_{polig}(n) + u_{diam}(n), \tag{7}$$

where n corresponds to a potential energy dependence on an even natural number $\mathbb{N}$ of point charges. In order to generalize, we begin by considering the case of five point charges as

indicated in Fig. 1. For this configuration, the number of terms contributing to each energy type are provided by equations (8), (9), and (10).

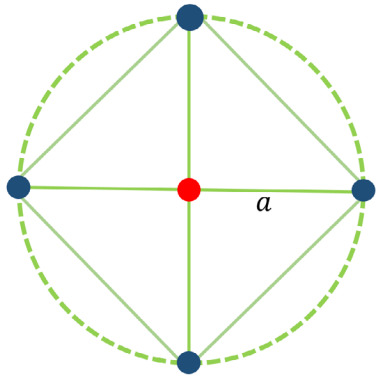

Figure 1. Top view of five point charges located at the vertices of a tetrahedron. The point charges in blue are located in the xy plane and inscribed into an imaginary circle of radius a. The point charge in red is located on the z axis along the axis of the circle.

$$u_{trid} = 4\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+h^2}}, \tag{8}$$

$$u_{polig} = 4\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a\sin\frac{\pi}{4}}, \tag{9}$$

$$u_{diam} = 2\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a}. \tag{10}$$

The terms corresponding to the u_{trid} energy result from taking the distance between point charge in the z axis and a charge in the plane by using the hypotenuse of the right triangle that is formed. The terms for the u_{polig} energy are obtained by accounting for all the inscribed diagonals that connect different vertices within the polygon. Since it is a regular polygon, it is possible to relate it to trigonometric functions based on the angles formed by the inscribed triangles within the same polygon. For the u_{diam} energy, it is sufficient to take the distance between pairs of charges that are diametrically opposed.

We can extend the system for $2n$ (with $n \in \mathbb{N}$) discrete point charges whose general equations are

$$u_{trid}(n) = n\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+h^2}}, \tag{11}$$

$$u_{polig}(n) = n\sum_{k=1}^{\frac{n}{2}-1}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a\sin\left(\frac{k\pi}{n}\right)}, \tag{12}$$

$$u_{diam}(n) = \frac{n}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a}. \tag{13}$$

We have taken an even number of charges since the inscribed polygons are symmetrical and allow for faster calculation. Moreover, for our purpose, symmetrical polygons converge to a circle for a very large number of charges n. To illustrate the mathematical induction process we include Table 1.

To better understand the energy contribution by each term in Eq. (7), we plot $u_{trid}(n)$, $u_{polig}(n)$ and $u_{diam}(n)$ in Fig. 2a. We see that the term $u_{polig}(n)$ contributes the most to

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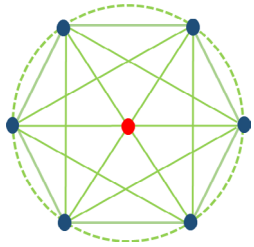
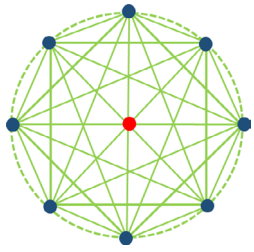
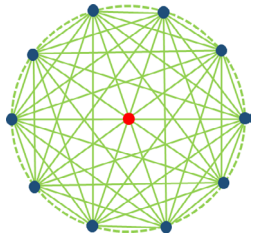
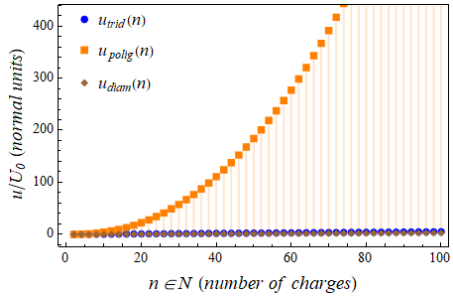
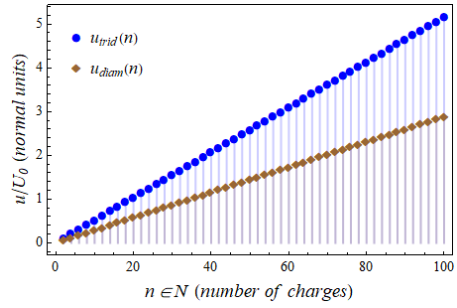
Charge configurations for $n = 6, 8, 10$	Energy contribution by the point charges located in the plane of the circle	Energy contribution by the point charge located in the axis of the circle at point $P(0,0,h)$
	$u_{polig} = 6\frac{q^2}{4\pi\varepsilon_0}\times \sum_{k=1}^{2}\frac{1}{2a\sin\left(\frac{k\pi}{6}\right)}$ $u_{diam} = 3\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a}$	$u_{trid} = 6\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+h^2}}$
	$u_{polig} = 8\frac{q^2}{4\pi\varepsilon_0}\times \sum_{k=1}^{3}\frac{1}{2a\sin\left(\frac{k\pi}{8}\right)}$ $u_{diam} = 4\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a}$	$u_{trid} = 8\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+h^2}}$
	$u_{polig} = 10\frac{q^2}{4\pi\varepsilon_0}\times \sum_{k=1}^{4}\frac{1}{2a\sin\left(\frac{k\pi}{10}\right)}$ $u_{diam} = 5\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2a}$	$u_{trid} = 10\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+h^2}}$

Table 1. Different circular charge configurations for $n = 6, 8, 10$ (in dark blue) and a point charge (in red) at height h on the axis of the circle.

the total potential energy $U_{Total}(n)$, and keeps increasing. In Fig. 2b, we plot $u_{trid}(n)$ and $u_{diam}(n)$ for a better comparison. Although the potential energy $U_{Total}(n)$ is apparently zero in the figure 3; when $n = 2$ corresponding to the lower case, the total potential energy value is of the order of 0.08, where we have taken $h = 2m$ and $a = 1m$.



(a)
Potential energy terms $u_{trid}(n)$, $u_{polig}(n)$ and $u_{diam}(n)$ in dimensionless units. The term $u_{polig}(n)$ contributes the most to the total potential energy $U_{Total}(n)$.



(b)
Potential energy terms $u_{trid}(n)$ and $u_{diam}(n)$ in dimensionless units.

Figure 2. Energy contribution to the potential energy as a function of even natural numbers n. The parameters have been rescaled and set to $q = |e^-|$, $h = 2m$ and $a = 1m$.

This discrete ring-type distribution allow us to classify the total amount of energy as the sum of three contributions. If we consider the interaction between all pairs of point charges, the $U_{Total}(n)$ diverges for large n as can be seen in Fig. 3. It is worth noticing that the increasing u_{polig} contribution is cut off in Fig. 2a but still remains increasing.

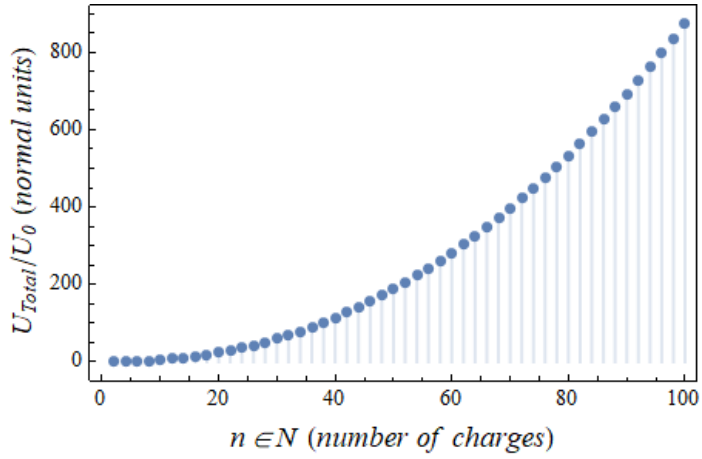


Figure 3. Normalized total potential energy $U_{Total}(n)$ as a function of even natural numbers n.

In Fig. 4a, a description of the energy behavior is shown as a function of the height h of the charge along the z-axis and the radius of the circle a where the polygon resulting from the distribution of discrete charges is inscribed. We observe that the energy variations are imperceptible with changes in height while keeping the radius fixed. Unlike the previous case, the energy of the discrete charge distribution with a fixed height and changing the radius of the circle where the geometric polygon of the charges is inscribed shows noticeable differences as the radius increases, as can be seen in Fig. 4b. We observe that as the charges move further apart from each other, in addition to moving away from the charge on the z-axis, the energy is expected to decrease.

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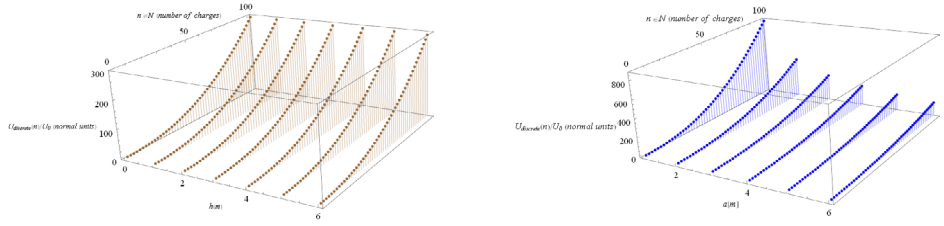
For the figures presented in this work, as well as for the calculations, we have used SI units, and the potential energy has been normalized by a factor of $U_0 = 0.5 \times 10^{27} J$, i. e., results are presented in normalized units of U/U_0. Units of height h, radius a and charges q are given in meters (m), meters (m) and Coulombs (C) units, respectively.



(a)
Total potential energy $U_{discrete}(n)$ as a function of even natural numbers n and height h for $h = 0, 1, 2, \ldots, 6m$ and with $a = 3m$.

(b)
Total potential energy $U_{discrete}(n)$ as a function of even natural numbers and radius a for $a = 0, 1, 2, \ldots, 6m$ and with $h = 4m$.

Figure 4. Description of the discrete arrangement of point charges as a function of height h, Fig. 4a, and radius a, Fig. 4b. We have set $q = \left|e^{-}\right|$ and all parameters have been rescaled to have normalized units.

The so called u_{trid} three dimensional energy contribution for the discrete case is compared with the following situations in next sections such as for a uni-dimensional continuous charged ring and a continuum charged torus. We observe that u_{polig} and u_{diam} correspond to the work needed to assemble the polygonal configuration of charges.

Electrostatic energy for a uniformly charged ring and a point charge

In this section, we consider a uniformly charged ring lying on the xy plane with a radius a and with a linear charge density λ. The ring's symmetry axis matches with the z axis and a point charge q is placed at a height h on the z axis.

The model is represented in Fig. 5 and the potential energy due to the system ring-point charge is calculated from

$$U = qV\left(\vec{r}_0\right), \tag{14}$$

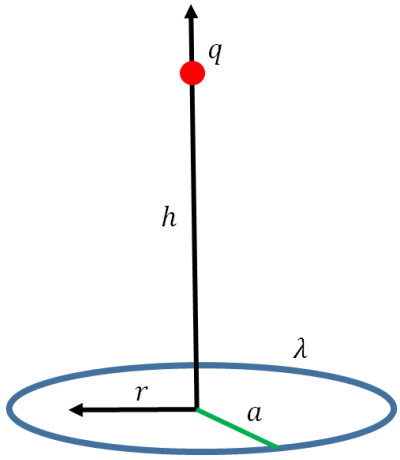


Figure 5. Geometrical configuration of an electrical charged ring of radius a and a point charge at height h on the ring's axis.

where the point charge q is located at point $\vec{r}_0 = (0,0,h)$, and the electric potential $V(\vec{r}_0)$ is due to a continuous charge distribution on the ring with radius a whose expression is the continuum limit of Eq. (14). Nevertheless, it is important to note that Eq. (14) does not represent the energy of the configuration as a whole, but specifically applies to the ring-point charge scenario.

Even though the calculation of the electric potential of the ring can be pursued by a simple integral, as done in many electromagnetism textbooks, here we cite results for the more general case off-axis and evaluate in the limit on-axis case. In (**Escalante**, 2021), a similar configuration but for a ring without axial symmetry was developed. The case of a ring symmetry in electric potential calculations is extended to include disc and cylinder generalizations in (**Charyyev** & **Shikakhwa**, 2018). We based our results on a previous work developed in (**Ciftja**, **Babineaux**, & **Hafeez**, 2009; **Good**, 1999). Even though the calculation for the case $r = 0$ is straightforward and can be found in any basic electrostatics textbook, we consider that the approach taken there significantly lacks details for further investigations. In (**Ciftja**, **Babineaux**, & **Hafeez**, 2009), the axially symmetric electrostatic potential for a uniformly charged ring $V(r,z)$ is given as

$$V(r,z) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda 4a}{\sqrt{(r+a)^2+z^2}}K\left(\frac{4ra}{(r+a)^2+z^2}\right), \tag{15}$$

where (r,z) are cylindrical coordinates, a is the ring's radius, and the azimuthal angle ϕ has been omitted. Here

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m\sin^2(\theta)}}, \tag{16}$$

is an elliptic integral of the first kind.

If we use equations (14) and (15) with $r = 0$ and $z = h$, the electric potential energy stored in the ring-point charge system is

$$U = qV(0,h). \tag{17}$$

With $\lambda = \frac{Q}{2\pi a}$, where Q is the total ring's charge, we can calculate the potential energy of the system U. In order to make a comparison with the discrete case discussed in the previous

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section, we assume that total ring's charge is an integer n multiplied by the fundamental electron charge, i.e., $Q = n\,|e^-|$, then the potential energy becomes

$$U = \frac{q}{2}\frac{n\,|e^-|}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+h^2}}, \tag{18}$$

where $K(0) = \frac{\pi}{2}$. Note that Eq. (18) matches exactly with Eq. (8) with $q = |e^-|$ and the corresponding graph of Eq. (18) should mirror $u_{trid}\,(n)$ in Fig. 2b. Even though these are different approaches to describe the ring-point charge configuration, we note that the results are equivalent; this is because in the limit of infinite charges in the polygonal configuration we obtain the continuous distribution of charge on the ring.

Electrostatic energy for a uniformly charged torus and a point charge

Another situation of interest and quite similar to those in the previous sections, corresponds to a uniformly charged torus and a point charge. Indeed, a torus is a geometrical configuration with circular symmetry, which in the appropriate limit, coincides with a charged ring. We consider a charge uniformly distributed on the surface of the torus, and calculate the electric potential at a point on its axial axis. The torus has a minor radius which is denoted by r and a major radius denoted by R (**Hernandes** & **Assis**, 2004; **Tashayev**, 2019). The torus is located on the xy plane and its axial symmetry axis coincides with the z axis as shown in Fig. 6.

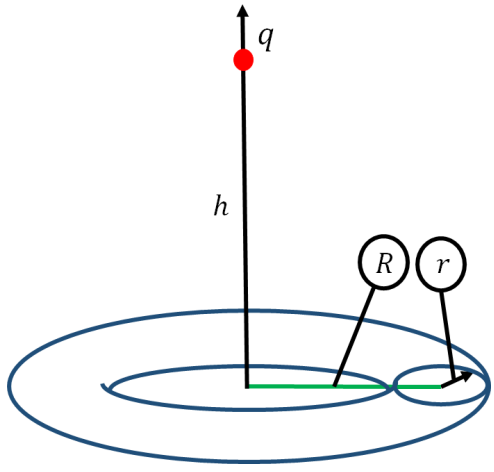

Figure 6. Torus with a uniform surface charge density and a punctual charge located at a height h on the axial z axis. The torus has a minor radius r and a major radius R.

The electric potential can be calculated using toroidal coordinates

$$x = a\frac{\sinh\eta\cos\varphi}{\cosh\eta-\cos\xi}, \tag{19}$$

$$y = a\frac{\sinh\eta\sin\varphi}{\cosh\eta-\cos\xi}, \tag{20}$$

$$z = a\frac{\sin\xi}{\cosh\eta-\cos\xi}. \tag{21}$$

The parameter a characterizes a ring with equations $x = a\cos\varphi$, $y = a\sin\varphi$, $z = 0$ as long as η goes to infinity. Here φ is the azimuthal angle. A constant parameter $\eta = \eta_0$, where $\eta_0 = \ln\left(\frac{R}{r} + \sqrt{\frac{R^2}{r^2} - 1}\right)$, describes the torus surface, thus, there are two regions: one internal identified with $\eta > \eta_0$, and one external with $\eta < \eta_0$. The electric potential for a uniform surface charge density on the torus in an exterior point, is given by

$$V(\eta < \eta_0, \xi, \varphi) = \sqrt{\cosh\eta - \cos\xi}\times \sum_{p=0}^{\infty} A_p \cos(p\xi) P_{p-\frac{1}{2}}(\cosh\eta), \tag{22}$$

where A_p are coefficients given by

$$A_p = \frac{\sqrt{2}A(2-\delta_{0p})\,Q_{p-\frac{1}{2}}(\cosh\eta_0)}{\pi P_{p-\frac{1}{2}}(\cosh\eta_0)}, \tag{23}$$

and δ_{ij} is the Kronecker delta, which is zero for $i \neq j$ and one for $i = j$. The functions $P_{p-\frac{1}{2}}(\cosh\eta_0)$ and $Q_{p-\frac{1}{2}}(\cosh\eta_0)$ are known as Legendre polynomials of the first and second kind, respectively. Eq. (22) was reported in (**Hernandes** & **Assis**, 2003; **Hernandes** & **Assis**, 2004) and we use it to calculate the potential energy for the system torus–point charge.

We are interested in the case when the minor radius r tends to zero, this is the situation when the torus corresponds to a charged ring. The limit $r \to 0$ is equivalent to choose $\eta_0 \gg 1$ and therefore $\cosh\eta_0 \gg 1$. Under this limit, we have $R \approx a$, and the coefficients in Eq. (23) are neglected except for $p = 0$, thus, the electric potential can be written as:

$$V(\eta,\xi,\varphi) = \frac{q}{4\pi\varepsilon_0\sqrt{2}a} \times \sqrt{\cosh\eta - \cos\xi}\,P_{-\frac{1}{2}}(\cosh\eta). \tag{24}$$

In spherical coordinates, the last expression becomes (**Hernandes** & **Assis**, 2004):

$$V(\rho,\theta,\varphi) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\left[(\rho^2-a^2)^2 + 4a^2\rho^2\cos^2\theta\right]^{\frac{1}{4}}}\times P_{-\frac{1}{2}}\left(\frac{\rho^2+a^2}{\left[(\rho^2-a^2)^2 + 4a^2\rho^2\cos^2\theta\right]^{\frac{1}{2}}}\right). \tag{25}$$

Using Eq. (5) together with Eq. (25), if we choose $\rho = h$, $\theta = 0$, and having $P_{-\frac{1}{2}}(1) = 1$, we obtain Eq. (18) which is exactly the same as for the charged ring, and for the discrete system (11). In this situation, the total amount of charge has been set to $q = n|e^-|$ for the charge on the surface of torus distributed uniformly, and we have omitted the corresponding plot since it coincides exactly with (18). Here we observe again that different approaches lead to equivalent results. Both cases, ring-point charge and torus-point charge in the zero minor radius limit coincide with the total potential energy contribution. This is reflected in expressions (18) and (25), under appropriate limits, and they match also with $u_{trid}(n)$ in Fig. 2b.

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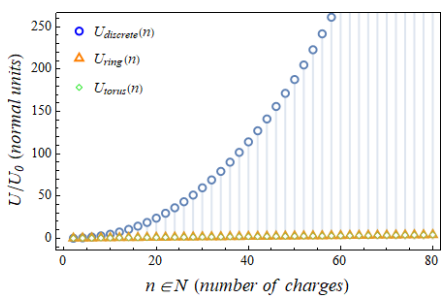
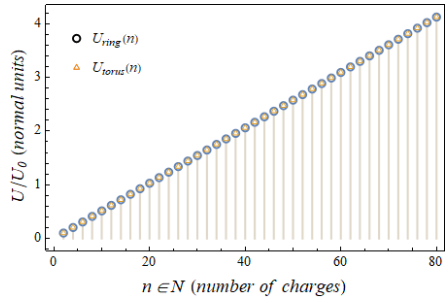


(a)
Electrostatic potential energies for the three cases studied, $U_{discrete}(n)$, $U_{ring}(n)$ and $U_{torus}(n)$ respectively, as a function of even natural numbers n. The potential energy for the discrete case, $U_{discrete}(n)$, grows at a higher rate than the other two cases involving continuous charge distributions.



(b)
Electrostatic potential energies $U_{ring}(n)$ and $U_{torus}(n)$. The two continuous charge distributions have a non-zero potential energy and grow at the same rate as n increases. Here we have considered a torus with zero minor radius limit which coincides with the charged ring.

Figure 7. Electric potential energies, in dimensionless units, as a function of even natural numbers n. The parameters have been rescaled and set to $q = |e^-|$, $h = 2m$, $a = 1m$, $R \approx a$.

Figures 7a and 7b show the electrostatic potential energies for the three cases considered in this work: a discrete charge configuration, a charged ring, and a charged torus. In Fig. 7a, it can be seen that the energy contribution of the discrete charge system is considerably higher when accounting for the interactions between the discrete charges located on the xy-plane. The energy contributions of the ring-point charge and the torus-point charge systems coincide due to the symmetry and assuming the minor radius r of the torus is very small. The potential energy contributions of the charged ring and torus are not visible in Fig. 7a, thus, the potential energy for these two systems has been plotted in Fig. 7b.

It is important to note that the self-energies of both the ring and the torus have not been considered in this case. According to the work of (**Ciftja**, 2023), the energy stored in the ring is divergent and independent of the amount of charge accumulated in it. This prevents us from making a comparison with the discrete case, which is based on the methodology we have applied across the three cases as a function of the number of charges involved.

Let us recall that, in the case of the ring and the torus, the continuous charge distribution can always be expressed as a multiple of the elementary charge of the electron, which allows for a comparison of the three schemes. In the discrete charge distribution, the number of charges considered is countable, which enables the calculation of a finite energy value for the charges that form the polygon. This, however, is not the case for the ring and, consequently, for the toroid in the limit where the minor radius r tends to zero. We can think of the ring and the torus as a single entity that interacts with a point charge, as in the cases studied in this work. The energy of the system is defined as that which arises from considering the interaction between the point charge located at an axial point and the electrostatic potential produced by the ring or the torus at the same point. This is the method that has been employed in the three schemes analyzed.

Concluding Remarks

In this paper, we used the concept of electrostatic potential energy for different arrangements of electric charge with circular symmetry. Firstly, we analyzed a discrete set of point charges arranged in a polygonal figure, this configuration corresponds to an inscribed polygon within a circle; as we increase the total amount of discrete charges we approximate

to a ring-type distribution of charges. This discrete character of the ring-type distribution allowed us to split the total amount of energy as the sum of three contributions. We considered the interaction between a pair of point charges and the total potential energy diverged, as we expected.

A similar analysis was made for a uni-dimensional continuous charged ring and a continuous charged torus. The total contribution of the electrostatic potential energy for the three arrangements on a point charge located on a symmetry axis perpendicular to the plane of the charge distributions becomes equivalent no matter the nature of the geometrical disposition, nevertheless, we take into account just the so called u_{trid} three dimensional energy contribution for the discrete case; the terms u_{polig} polygonal energy and u_{diam} diametrical energy correspond to the work needed to assemble the polygonal charge distribution and do not account of the charge-ring interaction. It is worth noticing that even though the last two cases were for continuous charge distributions, the total amount of charge preserves the quantization law of electrical charge, thus, this allowed us to obtain the equivalence between the three distributions.

We would like to emphasize also, that the term u_{diam} in Eq. (10), remains constant regardless of the number of charges, which could also be demonstrated in the case of a continuous distribution. Given the symmetry of the configurations, it is natural to expect this behavior for the ring and toroid in diametrically opposite locations. In the discrete case, we have directly identified the diametrical contributions between the corresponding pairs of point charges by constructing the regular polygon. However, in the continuous case, the process of integration over the respective charge differentials is carried out implicitly during the integration process. It is not immediately evident that the energetic contributions from diametrically opposite charge differentials can be separated from those of adjacent charge differentials along the arc length of the ring during the integration process. A similar argument applies to the case of the toroid. The authors believe that further investigation into these contributions could be explored in an extension or future work of this article.

We observe that even in classical books such as (**Feynman**, **Leighton**, & **Sands**, 1963; **Good**, 1999; **Griffiths**, 1999; **Kittel**, **Knight**, & **Ruderman**, 1973; **Wangsness**, 1997), electrostatic potential energy for continuous charge distributions are done as integrals around the entire space and few examples for the inside and outside volumetric charge distributions are given. In addition, the case of a uni-dimensional continuous charge distribution such as a ring-type shape is not clear. We hope that this work further clarifies such charge configurations. For the last two continuous cases, the total amount of energy contribution coincides as we considered a torus thin enough (in the limit $r \to 0$) for the exterior potential region equivalent to a uni-dimensional charged ring.

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Author Contributions

Conceptualization, writing, and original draft preparation, O. Aguilar−Loreto; methodology and investigation of examples and physics implied, images editing, B. Ordaz; formal analysis and writing, review and editing, A. Muñoz. All authors have read and agreed to the published version of the manuscript.

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Conflict of Interests

The authors declare that they have no affiliations with or involvement in any organization or entity with any financial interest in the subject matter or materials discussed in this manuscript

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