SOME FURTHER REMARKS ON THE LOCAL PATH-CONNECTEDNESS OF COMPACTIFICATIONS

by

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Resumen

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Se establecen algunos resultados sobre la conexión y la arco-conexión locales de compactaciones de espacios localmente conexos por adición de conjuntos infinitos. Los resultados depende de propiedades topológicas de estos últimos conjuntos y generalizan resultados previos de los autores.

Palabras clave: Compactificaciones, conexión y arco-conexión locales, componentes conexas y arco-conexas.

Abstract

Some results are presented on the local connectedness and local path-connectedness of compartifications of a topological space by addition of an infinite set of points. The results depend on topological properties of this latter set and generalize previousresults.

Key words: Compactifications, local connectedness and local path-connectedness, conected components and path-components

1. Introduction

If X is a topological space ([1], p. 13) and $\tilde{X}$ is a compact Hausdorff space ([1], p. 83, p. 93) containing X as an open dense suhset, $\tilde{X}$ is said to be a compatification of X by addition of the set N = $\tilde{X}$ - X. Observe that N is a closed subset of $\tilde{X}$. The space X is then Hausdorff and locally compact ([i], p. 102).

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In [2] it has been shown that if X is locally connected ([1], p. 129) and the set N is such that $D^{(n)}(N) = \emptyset$ for some $n \geq 0$ ($D^{(n)}(N)$ is defined inductively by $D^{(0)}(N) = N, D^{(1)}(N) = D(N)$ is the set of limit points of N, and $D^{(n)}(N) = D\left(D^{(n-1)}(N)\right)$, for $n \geq 1$) then, provided $\widetilde{X}$ has only finitely many connected components ([1], p. 127), also $\widetilde{X}$ is locally connected. Furthermore, under the same assumptions, if X is locally path-connected (see [2] for the appropriate definitions) and any point $\omega \in N$ has a countable fundamental system of neighborhoods in $\widetilde{X}$, then $\widetilde{X}$ is also locally path-connected. These results allowed to investigate in [2] the local connectedness and local path-connectedness properties of the compactifications of $\sigma-$compact spaces (Hausdorff spaces which are the union of countably many open subsets with compact closure) and of locally connected paracompact spaces (locally connected spaces in which the connected components are $\sigma-$compact).

As in [2], our main reference for generalities about topological and uniform spaces and their connectedness properties is [1]. See also [3]. A $\sigma-$compact space is called countable at infinity in [1], p. 106. In what follows, all topological spaces are supposed to be Hausdorff and the relevant compactifications are assumed to have only finitely many connected components (a compact space having infinitely many connected components can not be locally connected). For motivation about results on local connectedness and local path-connectedness of compactifications, see [2], [4], [5], [6], [7]. In this brief paper we examine results on these properties, mainly on local path- connectedness, which are motived by or extend results in [2] but, in spite of some interest in their own, are not as conclusive as them. They may, however, shorten proofs or make them more accessible.

2. On local connectedness

In this section we extend results in Section 4 of [2]. We begin by observing that the basic assumption of [2], that of the compactifying set $N = \widetilde{X} - X$ being such that $D^{(n)}(N) = \emptyset$ for some $n \geq 0$, points in the direction of a strong disconnectedness of this set. In fact, if $\omega \in N$ and $C_\omega(N)$ denotes the connected component of ω in N, also $D^{(n)}(C_\omega(N)) = \emptyset$, and if $1 \leq m \leq n$ is such that $D^{(m)}(C_\omega(N)) = \emptyset$ and $D^{(m-1)}(C_\omega(N)) \neq \emptyset$, the fact that $C_\omega(N)$ can not have isolated points, unless it reduces to a point, ensures that $m = 1$ and thus $C_\omega(N) = \{\omega\}$. In view of this, some results in [2], Section 4, can be considerably extended as we next show.

Observe that asserting that $\widetilde{X}$ is locally connected at $\omega \in N$ (i.e., that ω has a fundamental system of connected neighborhood in $\widetilde{X}$) is obviously equivalent to prove that for any open neighborhoods U of ω in $\widetilde{X}$, $\omega \notin \overline{\varepsilon_\omega(U)}$, where $\varepsilon_\omega(U) = U - C_\omega(U)$ (here and in what follows, $\overline{A}$, for $A \subseteq \widetilde{X}$, is the closure of A in $\widetilde{X}$;also, A° stands for the interior of A in $\widetilde{X}$).

Theorem 2.1 *Let X be locally connected and let $\widetilde{X}$ be a connected compactification of X by addition of a set N such that $C_\omega(N) = \{\omega\}$ for all $\omega \in N$. Then, $\widetilde{X}$ is locally connected.*

Proof. Let $\omega \in N$ and U, W be open neighborhoods of ω such that $\overline{W} \subseteq U$. Let $\varepsilon_\omega(U) = U - C_\omega(U)$ and $\varepsilon'_\omega(\overline{W}) = \varepsilon_\omega(U) \cap \overline{W}$. If $\omega \in \overline{\varepsilon_\omega(U)}$, also $\omega \in \overline{\varepsilon'_\omega(\overline{W})}$ (as W is a neighborhood of ω), and we will prove that the connected component C_ω of ω in $\overline{\varepsilon'_\omega(\overline{W})}$ meets $Fr(\overline{W}) := \overline{W} \cap \overline{\widetilde{X} - \overline{W}}$. In fact, let A be a clopen (both open and closed) subset of $\overline{\varepsilon'_\omega(\overline{W})}$ such that $C_\omega \subseteq A$. Since A is open and $\omega \in A$ then $A \cap \varepsilon'_\omega(\overline{W}) \neq \emptyset$, and a connected component C of $\overline{W}, C \subseteq \varepsilon'_\omega(\overline{W})$, should exist such that $A \cap C \neq \emptyset$. Since $A \cap \varepsilon'_\omega(\overline{W})$ is clopen in $\varepsilon'_\omega(\overline{W})$ then $C \subseteq A$, and now we prove that $C \cap Fr(\overline{W}) \neq \emptyset$. Assume not. Then, for each $y \in Fr(\overline{W})$, a clopen subset B_y of $\overline{W}$ should exist such that $C \subseteq B_y$ and $y \notin B_y$ ([1], p. 224, proposition 6). Thus $\{\overline{W} - B_y \mid y \in Fr(\overline{W})\}$is an open covering of $Fr(\overline{W})$, and $Fr(\overline{W})$ being compact, a finite number of them, $\overline{W} - B_{y_k}$, $k = 1, 2, 3, ..., n$, still covers $Fr(\overline{W})$. But then $B = \bigcap_{k=1}^n B_{y_k}$ would be clopen in $\overline{W}$ and $B \cap Fr(\overline{W}) = \emptyset$, so that $B \subseteq \overline{W}^\circ$. This is absurd, since then B would be clopen in $\widetilde{X}$, which is assumed to be connected. Then $C \cap Fr(\overline{W}) = \emptyset$, therefore $A \cap Fr(\overline{W}) = \emptyset$, and ([1], p. 224, proposition 6) since C_ω is the intersection of the clopen subsets of $\overline{\varepsilon'_\omega(\overline{W})}$ in which it is contained (a family of closed subsets of $\widetilde{X}$ with the finite intersection property), also $C_\omega \cap Fr(\overline{W}) \neq \emptyset$. Now we prove that $C_\omega \cap (U - N) = \emptyset$. Assume not, and let $x \in C_\omega \cap (U - N)$ and C be the connected component of x in $U - N$. Then $C \cap C_\omega \neq \emptyset$, so that $C \cap C_\omega(U) \neq \emptyset$, and since C is a connected subset of U, also $C \subseteq C_\omega(U)$. Then $C \cap \varepsilon_\omega(U) = \emptyset$, and since in addition C is open, $C \cap \overline{\varepsilon_\omega(U)} = \emptyset$, so that $C \cap \overline{\varepsilon'_\omega(\overline{W})} = \emptyset$. This is contradictory. Hence $C_\omega \subseteq N$. But this is again contradictory, because then there would be $\omega' \in C_\omega \cap Fr(\overline{W})$, so that $\omega' \in N$

and $\omega' \neq \omega$, while $C_\omega \subseteq C_\omega(N) = \{\omega\}$. Therefore, $\omega \notin \overline{\varepsilon_\omega(U)}$. □

The above proof follows closely ideas in [4] and is a refinement of the proof of Lemma 4.1 in [2].

Remark 2.1 The result in Theorem 2.1 still holds if $\widetilde{X}$ is not connected but has instead, as previously mentioned, finitely many connected components. Just replace $\widetilde{X}$ by $C_\omega\left(\widetilde{X}\right)$ in the above proof.

Remark 2.2 If U is open, the set $\varepsilon_\omega(U) := U - C_\omega(U)$ is identical with the union $\widetilde{\varepsilon_\omega}(U)$ of the components C of U such that $\omega \notin \overline{C}$. In fact, if $x \in \varepsilon_\omega(U)$ and $C = C_x(U)$ then $\omega \notin \overline{C}$, since, on the contrary, $C \cup \{\omega\}$ would be connected and thus $C \cup \{\omega\} \subseteq C_\omega(U)$; on the other hand, if $x \in \widetilde{\varepsilon_\omega}(U)$ then x belongs to a component C of U such that $\omega \notin \overline{C}$, so that $C \neq C_\omega(U)$; hence, $x \notin C_\omega(U)$.

3. On local path-connectedness.

Now we extend Lemma 5.1 of [2]. First we observe that the condition $\omega \notin \overline{\varepsilon_\omega(U)}$ for any neighborhood U of $\omega \in N$, which is necessary and sufficient for local connectedness at ω, does not seem to be related to local path-connectedness, even when ω has countable fundamental systems of neighborhoods (see [2], Remark 5.5). More relevant for such purpose seems to be the set $\Im_\omega(U)$, which is the union of the connected components C of $U - N$ such that $\omega \notin \overline{C}$. The definition of $\Im_\omega(U)$ is motivated by what was stated in Remark 2.2. We observe that for the choice of U in the proofs of [2], Lemmas 4.1 and 5.1, it is obvious that $\varepsilon_\omega(U) = \Im_\omega(U)$. The following three lemmas hold.

Lemma 3.1 *If U is an open neighborhood of $\omega \in N$ then $\varepsilon_\omega(U) \cap X \subseteq \Im_\omega(U)$ and $\overline{\varepsilon_\omega(U)} \subseteq \overline{\Im_\omega(U)}$.*

Proof. Any $x \in \varepsilon_\omega(U)$ belongs to a connected component C of U such that $\omega \notin \overline{C}$. On the other hand, x is in a connected component C' of $U - N$, and since $U - N \subseteq U$, also $C' \subseteq C$, so that $\omega \notin \overline{C'}$; thus $C' \subseteq \Im_\omega(U)$, which implies that $x \in \Im_\omega(U)$. Now let $\omega \in \overline{\varepsilon_\omega(U)}$ and V be an open neighborhood of ω in $\widetilde{X}$. Then $V \cap \varepsilon_\omega(U) \neq \emptyset$. Since $\varepsilon_\omega(U) = U - C_\omega(U)$ is open in U and therefore in $\widetilde{X}$, $(V \cap \varepsilon_\omega(U)) \cap X = V \cap (\varepsilon_\omega(U) \cap X)$ is non–empty and contained in $V \cap \Im_\omega(U)$. Therefore $V \cap \Im_\omega(U) \neq \emptyset$, and $\omega \in \overline{\Im_\omega(U)}$. □

Lemma 3.2 *Assume X to be locally path-connected and that for $\omega \in N$ and any open neighborhood U of ω, $\omega \notin \overline{\Im_\omega(U)}$. Then, $\widetilde{X}$ is locally connected at ω. Furthermore, if ω also has a countable fundamental system of neighborhoods, and U is an open neighborhood of ω, any point $a \in U - \overline{\Im_\omega(U)}$, $a \in X$, can be joined to ω by means of a path in $U - \overline{\Im_\omega(U)}$.*

Proof. The first assertion follows from Lemma 3.1. Now let ω and U be as above and let $(V_n)_{n\geq 1}$ be a countable fundamental system of neighborhoods of ω. We may assume $V_{n+1} \subseteq V_n \subseteq U - \overline{\Im_\omega(U)}$ for all $n \geq 1$. For $a = a_o \in \left(U - \overline{\Im_\omega(U)}\right) \cap X$ there is $a_1 \in V_1 \cap C_{a_o}(U - N)$ such that $\omega \in \overline{C_{a_1}(V_1 - N)}$; if not $V_1 \cap C_{a_o}(U - N) \subseteq \Im_\omega(V_1)$, and for some neighborhood $W_1 \subseteq V_1$ of ω such that $W_1 \cap \Im_\omega(V_1) = \emptyset$, also $W_1 \cap C_{a_o}(U - N) = \emptyset$, which, since $\omega \in \overline{C_{a_o}(U - N)}$, is absurd. The same argument shows for some $a_2 \in V_2 \cap C_{a_1}(V_1 - N)$ that $\omega \in \overline{C_{a_2}(V_2 - N)}$, and iteration allows, with $V_0 = U$, to obtain a sequence $(a_n)_{n\geq 0}$ such that $a_{n+1} \in V_{n+1} \cap C_{a_n}(V_n - N)$ and $\omega \in \overline{C_{a_n}(V_n - N)}$ for all $n \geq 0$. Since X is locally path- connected, components and paths-components of X coincide. Then, a sequence $0 = t_o < t_1 < \cdots < t_n < \cdots < 1$, $t_n \to 1$, and paths $\alpha_n : [t_n, t_{n+1}] \to C_{a_n}(V_n - N)$, $n \geq 0$, exist such that $\alpha_n(t_n) = a_n, \alpha_n(t_{n+1}) = a_{n+1}$. Observe that $\alpha_n([t_n, t_{n+1}]) \subseteq V_m$ for $m \geq 0$ and $n \geq m$, so that $\alpha : [0,1] \to U$, defined by $\alpha(t) = \alpha_n(t)$ if $t_n \leq t \leq t_{n+1}$ and $\alpha(1) = \omega$, actually is a path in U joining a_o and ω. As a matter of fact, $\alpha(t) \in V_n \subseteq U - \overline{\Im_\omega(U)}$ for $n \geq 1$ and $t \geq t_n$, and since $\alpha([t_o, t_1]) \subseteq C_{a_o}(U - N)$, $C_{a_o}(U - N)$ is open in U and $\overline{C_{a_o}(U - N)} \cap \Im_\omega(U) = \emptyset$, also $\alpha([t_o, t_1]) \subseteq U - \overline{\Im_\omega(U)}$. Thus, α is a path in $U - \overline{\Im_\omega(U)}$. □

Lemma 3.3 *Let ω and U be as in the previous lemma. Then, any point $\omega' \in U_o = U - \overline{\Im_\omega(U)}$, $\omega' \in N$, such that $\omega' \notin \overline{\Im_\omega(V)}$ for any open neighborhood V of ω' in $\widetilde{X}$, and admitting a countable fundamental system of neighborhoods, can be joined to ω by a path in U_o.*

Proof. Any point in $U_o \cap X$ can be so joined to ω, as shown in the previous lemma. Now assume that the assumptions hold for $\omega' \in U_o \cap N$. Again, from Lemma 3.2 it follows that any point $x \in U_1 = U_o - \overline{\Im_{\omega'}(U_o)}$, $x \in X$, can be joined to ω' by a path in U_1. Thus, since $U_1 \subseteq U_o$, if $U_1 \cap X \neq \emptyset$, the assertion will follow. But since U_1 is a neighborhood of ω' and ω' is in the closure of X, this is trivial. □

The above results imply the following theorem.

Theorem 3.1 *Let $\widetilde{X}$ be a compactification of the locally path- connected space X by addition of the set*

N. Assume that any point $\omega \in N$ has a countable fundamental system of neighborhoods and that for any open neighborhood U of ω, $\omega \notin \overline{\Im_\omega(U)}$. Then $\widetilde{X}$ is locally path-connected.

Proof. Clearly $\widetilde{X}$ is locally connected , and from the previous lemma it follows that for any $\omega \in N$ and any open neighborhood U of $\omega, U - \overline{\Im_\omega(U)}$ is a path-connected neighborhood of ω. □

Remark 3.1 The proof of Theorem 3.1 follows ideas in [5] . Let $\widetilde{X}$ be a compactification of X by addition of N. A subset M of N is called *thin* if it is open in N and any $\omega \in M$ has a fundamental system $(V_n)_{n\geq 0}$ of open neighborhoods such that $\omega \notin \overline{\Im_\omega(V_n)}$ for all $n \geq 0$. We may assume that $V_n \cap (N - M) = \emptyset$,and it follows that if X is locally path-connected then $V_n - \overline{\Im_\omega(V_n)}$, $n \geq 0$, is a fundamental system of open path connected neighborhoods of ω, so that $X \cup M$ is locally path-connected. Clearly $X \cup M$ is open and dense in $\widetilde{X}$, and therefore $\widetilde{X}$ is a compactification of $X \cup M$ by addition of $N - M$.

Remark 3.2 If $\widetilde{X}$ is a compactification of the locally path-connected space X by addition of N and if M is an open subset of N such that each $\omega \in M$ has a countable fundamental system $(V_n)_{n\geq 0}$ of open neighborhoods with $V_n - N$ having, for each n, only finitely many components, then M is thin. In fact, we may assume that $V_n \cap (N - M) = \emptyset$ for all n, and if $C_1, C_2, ..., C_m$ are the components of $V_n - M$ such that $\omega \notin \overline{C_k}$, $k = 1, 2, ..., m$, then $\omega \notin \overline{\Im_\omega(V_n)} = \overline{C_1} \cup ... \cup \overline{C_m}$.

In view of Remark 3.1 and Corollary 5.4 in [2] , the following corollary of Theorem 3.1 holds.

Corollary 3.1 *If $\widetilde{X}$ is a compactification of the locally path-connected space X by addition of the set N, if every point ω of N has a countable fundamental system of neighborhoods, and if M is a thin subset of N such that $D^{(n)}(N - M) = \emptyset$ for some $n \geq 0$, then $\widetilde{X}$ is locally path- connected.*

The following example has some interesting features.

Example 3.1 Let

$$X = \{(x, n/x) \mid x \in R, x \neq 0, n \in Z, n \neq 0\}$$

be endowed with the topology of subspace of R^2. Also let $\widetilde{X} = X \cup (\{0\} \times [-\infty, +\infty]) \cup ([-\infty, +\infty] \times \{0\})$ be given the topology of subspace of $[-\infty, +\infty] \times [-\infty, +\infty]$.

Then $\widetilde{X}$ is a compactification of X by addition of the set $N = (\{0\} \times [-\infty, +\infty] \cup ([-\infty, +\infty] \times \{0\})$. Clearly $M = (\{0\} \times R) \cup (R \times \{0\})$ is a thin subset of N and

$$N - M = \{(0, -\infty), (0, +\infty), (-\infty, 0), (+\infty, 0)\}$$

is finite, so that from Corollary 3.1, $\widetilde{X}$ is locally connected and locally path-connected. Observe however that $C_\omega(N) = N$ for each $\omega \in N$ and therefore N is not totally disconnected. This example also shows that neither condition $\omega \notin \overline{\varepsilon_\omega(U)}$ or $\omega \notin \overline{\Im_\omega(U)}$ implies total disconnectedness of N, and that $\widetilde{X}$ can still be locally connected or locally path-connected when this condition fails to hold for N.

Remark 3.3 We do not know as yet whether the only conditions of having countable fundamental systems of neighborhoods at $\omega \in N$ and of $\omega \notin \overline{\varepsilon_\omega(U)}$ for any neighborhood U of ω can guaranty the local path -connectedness of $\widetilde{X}$ at ω.

Remark 3.4 It is readily seen that if the space X in Example 3.1 is given the uniform structure $\mathcal{U}$ ([1], p. 181) of subspace of $[-\infty, +\infty] \times [-\infty, +\infty]$ and $x \in X$ then $X = A_x$, where A_x is the set of points of X which can be joined to x by a V-chain ([1], p. 224) for all $V \in \mathcal{U}$. Observe, however, that X is not connected, as the case would be if X were compact ([1], p. 224, proposition 6), a result that was crucial in all considerations above.

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