THE STONE-ČECH COMPACTIFICATION IN THE CATEGORY OF SHEAVES OF SETS

by

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Abstract

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We consider the Category of Sheaves of Sets. The morphisms are chosen in such a way that a change of the base space is allowed via continuous functions. Following M. M. Clementino, E. Giuli and W. Tholen in "Topology in a Category: Compactness", we define a proper $(\varepsilon, \mathcal{M})$-factorization system for morphisms and a closure operator with respect to $\mathcal{M}$. The Stone-Čech compactification is defined for any sheaf (E, p, T) of sets by adapting standard germination processes to construct a sheaf over the Stone-Čech compactification $\beta(T)$ of T. We prove that the sheaf constructed satisfies a suitable universal property characterizing the Stone-Čech compactification of a sheaf of sets.

Key words: Stone-Čech compactifications, Sheaves.

Resumen

Se considera la categoría de haces de conjuntos en la cual se han escogido los morfismos de tal manera que se puedan realizar cambios del espacio base por vía de funciones continuas. Siguiendo la teoría de M.M. Clementino, E. Giuli y W. Tholen, expuesta en "Topology in a Category: Compactness", se define un sistema propio $(\varepsilon, \mathcal{M})$ de factorización de morfismos y un operador de clausura con respecto a $\mathcal{M}$. Se define el compactado de Stone-Čech de un haz arbitrario (E, p, T) utilizando procesos clásicos de germinación para construir un haz de conjuntos sobre el compactado de Stone-Čech $\beta(T)$ de T. Se demuestra que el haz construido satisface la propiedad universal que caracteriza el compactado de Stone-Čech de un haz de conjuntos.

Palabras clave: Compactado de Stone-Čech, haces.

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1. Preliminaries

Let E and T be topological spaces, $p : E \to T$ be a surjective function. The triple (E, p, T) is by definition a *sheaf of sets* provided that p is a local homeomorphism, that is each point $a \in E$ has an open neighborhood which is mapped homeomorphically by p onto an open subset of T. Let $t \in T$, the set

$$E_t = p^{-1}(t) = \{a \in E : p(a) = t\}$$

is called the *fiber above* t. Note that $E = \coprod_{t \in T} E_t$. A *local selection* for p is a function $\sigma : U \to E$ such that $U \subset T$ is an open set and $p \circ \sigma$ is the identity map of U. A *local section for* p is by definition a continuous selection. Denote by $\Gamma_U(p) = \{\sigma : U \to E \mid p \circ \sigma = \mathrm{Id}_U\}$. If $U = T$, σ is a *global section* and we write $\Gamma(p)$ instead of $\Gamma_T(p)$. A set Σ of local sections is called *full* if for every $x \in E$ there exists a section $\sigma \in \Sigma$ such that $\sigma(p(x)) = x$.

From the definitions it readily follows:

(1) If σ and τ are sections such that $\sigma(t) = \tau(t)$, for some $t \in \mathrm{Dom}\,\sigma \cap \mathrm{Dom}\,\tau$, then there exists an open neighborhood V of t such that $\sigma \restriction_V = \tau \restriction_V$.

(2) The collection of the ranges of the local sections for p form a basis for the topology of E.

(3) The topology induced by E on each fiber is the discrete topology.

Now we state a well known existence theorem for sheaves of sets whose proof is elementary and direct.

Existence Theorem. *Let T be a topological space, $p : E \to T$ be a surjective function and Σ a full set of selections for p such that if σ, $\tau \in \Sigma$ and $\sigma(t) = \tau(t)$ then there exists a neighborhood V of t such that $\sigma(s) = \tau(s)$ for each $s \in V$. Then the sets $\sigma(U)$ where $\sigma \in \Sigma$ and U is an open set contained in the domain of σ, form a basis for a topology on E such that (E, p, T) is a sheaf of sets and each $\sigma \in \Sigma$ is a section for p.*

A germination process with change of the base space (via a continuous function) can be formulated now. By means of this process, sheaves of sets will be constructed in terms of data provided by two topological spaces T and S, a set of global selections for a surjective function $p : E \to T$ and a continuous function $\varphi : T \to S$.

Theorem (Germination Process). *Let T and S be topological spaces. Suppose that $\varphi : T \to S$ is a continuous function, $p : E \to T$ is a surjective function and Σ is a set of global selections for p. For each $s \in S$ denote by R_s the equivalence relation in Σ defined by $\sigma R_s \tau$ if and only if there is a neighborhood V of s such that $\sigma \restriction_{\varphi^{-1}(V)} = \tau \restriction_{\varphi^{-1}(V)}$. Let $\widehat{E}$ be the disjoint union of the family $\{\Sigma/R_s : s \in S\}$, $\widehat{p} : \widehat{E} \to S$ the function defined by $\widehat{p}([\sigma]_s) = s$ and $[\sigma]_s$ the equivalent class of σ module R_s. Then the triple $(\widehat{E}, \widehat{p}, S)$ is a sheaf of sets and for each $\sigma \in \Sigma$ the function $\widehat{\sigma} : S \to \widehat{E}$ defined by $\widehat{\sigma}(s) = [\sigma]_s$ is a section for $\widehat{p}$.*

The proof of this theorem is straighforward and is omitted. For a generalization of this result in the context of uniform bundles we refer to [5]. The triple $(\widehat{E}, \widehat{p}, S)$ is called *the sheaf constructed by germination from* p, Σ *and* φ.

Remark. In the set up of the precedent theorem if $s \notin \overline{\varphi(T)}$ then the fiber $\widehat{E}_s$ above s reduces to zero.

2. The Category of Sheaves of Sets

Denote by $\mathfrak{S}$ the Category of Sheaves of Sets. The objects of $\mathfrak{S}$, are sheaves of sets containing a full set of global sections. Let (E, p, T) and (F, q, S) be sheaves of sets, consider a continuous function $\ell : T \to S$ and let

$$\begin{aligned} F_\ell &= \{(t, m) \in T \times F : \ell(t) = q(m)\} \\ &= \bigcup \{\{t\} \times F_{\ell(t)} : t \in T\} \end{aligned}$$

and π_1 be the first projection function. Endow F_ℓ with the topology induced by the product topology of $T \times F$. It is apparent that the diagram

$$\begin{array}{ccc} F_\ell & \xrightarrow{\pi_2} & F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle q} \\ T & \xrightarrow[\ell]{} & S \end{array}$$

is a pullback in the category of topological spaces and continuous functions. Furthermore the triple (F_ℓ, π_1, T) is a sheaf of sets, in fact, if $(t, m) \in F_\ell$ and $\alpha \in \Gamma(q)$ is a global section such that $\alpha(\ell(t)) = m$ then $Q = (T \times \alpha(S)) \cap F_\ell$ is an open neighborhood of (t, m) and $\pi_1 \restriction_Q$ is an homeomorphism of Q onto T.

A morphism $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ in $\mathfrak{S}$ consists of a pair of continuous functions $\ell : T \to S$ and $\Lambda : F_\ell \to E$ such that $p\Lambda = \pi_1$. Note that if $\alpha \in \Gamma(q)$ then the function $\alpha_\Lambda : T \to E$ defined by $\alpha_\Lambda(t) = \Lambda(t, \alpha(\ell(t)))$ is a global section for p. Let (E, p, T), (F, q, S) and (G, ρ, R) be sheaves of sets and let $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ and $(\ell_1, \Lambda_1) : (F, q, S) \to (G, \rho, R)$ be morphisms. Define the morphism

$$(\ell_1, \Lambda_1) \circ (\ell, \Lambda) : (E, p, T) \to (G, \rho, R)$$

by $(\ell_1, \Lambda_1) \circ (\ell, \Lambda) = (\ell_1 \ell, \Delta)$, where $\Delta : G_{\ell_1 \ell} \to E$ is given by $\Delta(t, a) = \Lambda(t, \Lambda_1(\ell(t), a))$ for each $(t, a) \in G_{\ell_1 \ell}$.

The continuity of Δ follows directly from the continuity of the functions $(t,a) \mapsto (\ell(t),a) : G_{\ell_1\ell} \to G_{\ell_1}$, $(t,a) \mapsto \Lambda_1(\ell(t),a) : G_{\ell_1\ell} \to F$, $(t,a) \mapsto (t, \Lambda_1(\ell(t),a)) : G_{\ell_1\ell} \to F_\ell$ and from the continuity of Λ.

3. Product of sheaves of sets

Let $(E_i, p_i, T_i)_{i\in\Im}$ be a not empty family of sheaves of sets each one of them containing a full set of global sections. Without loss of generality it can be assumed that the $(T_i)_{i\in\Im}$ are pairwise disjoint. Let $T = \prod_{i\in\Im} T_i$ be the product space of the family $\{T_i\}_{i\in\Im}$ and $\pi_i : T \to T_i$ the i-proyection. For each $t = (t_i)_{i\in\Im} \in T$ consider $E_t = \coprod_{i\in\Im}(E_i)_{t_i}$ the disjoint union of the family $\{(E_i)_{t_i} : i \in \Im\}$, where $(E_i)_{t_i}$ denotes the fiber above t_i in the sheaf (E_i, p_i, T_i), and let $E = \coprod_{t\in T} E_t$ the disjoint union of the family $\{E_t : t \in T\}$. Define $p : E \to T$ by $p(t,a) = t$.

If $i \in \Im$ and α_i is a global section for p_i then the function $\alpha^i : T \to E$ defined by

$$\alpha^i(t) = (t, (t_i, \alpha_i(t_i)))$$

is a global selection for p.

Let $\Sigma = \{\alpha^i : \ \alpha_i \in \Gamma(p_i) \ \ \& \ \ i \in \Im\}$. Suppose that α^i, $\beta^j \in \Sigma$, $t = (t_i)_{i\in\Im} \in T$ and $\alpha^i(t) = \beta^j(t)$. Since $(t_i, \alpha_i(t_i)) = (t_j, \beta_j(t_j))$ then $t_i = t_j$ and $\alpha_i(t_i) = \beta_j(t_j)$, this implies that $i = j$, thus there exists a neighborhood V of t_i in T_i such that $\alpha_i(r_i) = \beta_i(r_i)$ for each $r_i \in V$. If $s \in T$ is such that $s_i \in V$ then $\alpha^i(s) = (s, (s_i, \alpha_i(s_i))) = (s, (s_i, \beta_i(s_i))) = \beta^i(s)$. The Existence Theorem guarantees that the ranges of the restrictions of elements of Σ to open sets of T form a basis for a topology in E such that (E, p, T) is a sheaf of sets and each element of Σ is a section.

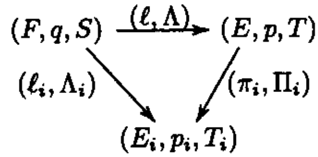
Now consider $i \in \Im$, the sheaf of sets (E_i, p_i, T_i), the continuous projection $\pi_i : T \to T_i$ and the function $\Pi_i : (E_i)_{\pi_i} \to E$, where

$$(E_i)_{\pi_i} = \bigcup_{t\in T} \{t\} \times (E_i)_{\pi_i(t)} ,$$

defined by $\Pi_i(t,a) = (t,(t_i,a))$.

The function Π_i is continuous, in fact, a basic neighborhood W of $(t,(t_i,a))$ is the range of the restriction to an open neighborhood V of t, of a global section $\alpha^i \in \Sigma$ for p such that $\alpha_i(t_i) = a$. If $r \in V$ and if $b = \alpha_i(\pi_i(r))$ then $\Pi(r,b) = (r,(r_i,b)) \in W$.

Hence the pair $(\pi_i, \Pi_i) : (E,p,T) \to (E_i, p_i, T_i)$ is a morphism in $\mathfrak{S}$. Furthermore if (F,q,S) is a sheaf of sets with a full set of global sections and $(\ell_i, \Lambda_i) : (F,q,S) \to (E_i, p_i, T_i)$ is a morphism for each $i \in \Im$ then there exists a unique morphism $(\ell, \Lambda) : (F,q,S) \to (E,p,T)$ such that the diagram



commutes.

The function ℓ is the only one of its kind because T is the product of the family of topological spaces $(T_i)_{i\in\Im}$. The function Λ is defined by $\Lambda(s,(t,(t_i,a))) = \Lambda_i(s,a)$.

Now we describe the monomorphisms and the epimorphisms in $\mathfrak{S}$.

4. Monomorphisms and epimorphisms

We characterize the monomorphisms of $\mathfrak{S}$ as follows.

Proposition 1. *Let (E,p,T) and (F,q,S) be sheaves of sets with full sets of global sections. The morphism $(\ell,\Lambda) : (E,p,T) \to (F,q,S)$ is a monomorphism if and only if ℓ is injective and Λ is surjective.*

Proof. Let $(\ell,\Lambda) : (E,p,T) \to (F,q,S)$ be a morphism such that ℓ is injective and Λ is surjective. Consider

$$(\ell_1, \Lambda_1) : (E', p', T') \to (E, p, T)$$

and

$$(\ell_2, \Lambda_2) : (E', p', T') \to (E, p, T)$$

two morphisms of $\mathfrak{S}$ such that

$$(\ell\,\ell_1, \Delta_1) = (\ell,\Lambda) \circ (\ell_1,\Lambda_1) = (\ell,\Lambda) \circ (\ell_2,\Lambda_2) = (\ell\,\ell_2, \Delta_2),$$

then $\ell\,\ell_1 = \ell\,\ell_2$ and hence $\ell_1 = \ell_2$ because ℓ is injective.

We have that $\Lambda_1(t', \Lambda(\ell_1(t'),a)) = \Lambda_2(t', \Lambda(\ell_2(t'),a))$ for each $(t',a) \in F_{\ell\ell_1} = F_{\ell\ell_2}$. Let $(t',b) \in E_{\ell_1} \subset T' \times E$ and $(t,a) \in F_\ell$ such that $\Lambda(t,a) = b$. Then $t = p(\Lambda(t,a)) = p(b) = \ell_1(t')$. Thus $b = \Lambda(\ell_1(t'),a) = \Lambda(\ell_2(t'),a)$, hence $\Lambda_1(t',b) = \Lambda_2(t',b)$, then $\Lambda_1 = \Lambda_2$ and it follows that (ℓ,Λ) is a monomorphism.

Conversely, let

$$(\ell,\Lambda) : (E,p,T) \to (F,q,S)$$

be a monomorphism. If T' is a topological space and $id_{T'} : T' \to T'$ is the identity function then the triple $(T', id_{T'}, T')$ is a sheaf of sets.

Let $j_1,\ j_2 : T' \to T$ be continuous functions such that $\ell\, j_1 = \ell\, j_2$. We claim that $j_1 = j_2$.

The pairs $(j_1, \Delta_1) : (T', id_{T'}, T') \to (E, p, T)$ and $(j_2, \Delta_2) : (T', id_{T'}, T') \to (E, p, T)$, where $\Delta_1(t', a) = t'$ and $\Delta_2(t', a) = t'$ are morphisms in $\mathfrak{S}$. Furthermore $(\ell, \Lambda) \circ (j_1, \Delta_1) = (\ell, \Lambda) \circ (j_2, \Delta_2)$, in fact, if $(\ell, \Lambda) \circ (j_1, \Delta_1) = (\ell j_1, L_1)$ then

$$L_1(t', a) = \Delta_1(t', \Lambda(j_1(t'), a)) = t'$$

for each $(t', a) \in F_{\ell j_1}$ and if $(\ell, \Lambda) \circ (j_2, \Delta_2) = (\ell j_2, L_2)$ then $L_2(t', a) = \Delta_2(t', \Lambda(j_2(t'), a)) = t'$ for each $(t', a) \in F_{\ell j_2}$. The assumption that (ℓ, Λ) is a monomorphism implies that $(j_1, \Delta_1) = (j_2, \Delta_2)$. Then $j_1 = j_2$ and consequently ℓ is a monomorphism in the category of topological spaces and continuous functions. This proves that ℓ is injective.

Now suppose that Λ is not surjective. There exists $x \in E$ such that $x \notin \Lambda(F_\ell)$. Consider the sheaf of sets (E', p', T') where $T' = \{t\}$ where $t = p(x)$, $E' = \{0, 1\}$, $p' : E' \to T'$ is defined by $p'(y) = 0$ for $y = 0,\ 1$ and both T' and E' have the discrete topology. Let $\ell' : T' \to T$ be the function defined by $\ell'(t) = t = p(x)$ and let $\Lambda_1,\ \ \Lambda_2 : E_{\ell'} \to E'$ be the functions defined by $\Lambda_1(t, a) = 0$ for each $a \in E_t$, $\Lambda_2(t, a) = 0$ if $a \neq x$ and $\Lambda_2(t, x) = 1$. We have that (ℓ', Λ_1) and (ℓ', Λ_2) are morphisms defined from (E', p', T') into (E, p, T) such that $(\ell, \Lambda) \circ (\ell', \Lambda_1) = (\ell, \Lambda) \circ (\ell', \Lambda_2)$. Since (ℓ, Λ) is a monomorphism we have $(\ell', \Lambda_1) = (\ell', \Lambda_2)$. Thus Λ is surjective. □

Proposition 2. *Let (E, p, T) and (F, q, S) be sheaves of sets each one with full sets of global sections. Let $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ be a morphism such that $\ell : T \to S$ is surjective and $\Lambda : F_\ell \to E$ is injective. Then (ℓ, Λ) is an epimorphism.*

Proof. Let (F', q', S') be a sheaf of sets with a full set of global sections. Consider

$$(\ell_1, \Lambda_1)\ ,\ (\ell_2, \Lambda_2) : (F, q, S) \to (F', q', S')$$

morphisms such that

$$(\ell_1 \ell, \Delta_1) = (\ell_1, \Lambda_1) \circ (\ell, \Lambda) = (\ell_2, \Lambda_2) \circ (\ell, \Lambda) = (\ell_2 \ell, \Delta_2).$$

Since ℓ is surjective and $\ell_1 \ell = \ell_2 \ell$ then $\ell_1 = \ell_2$. We claim that $\Lambda_1 = \Lambda_2$.

Taking into account that

$$\Lambda(t, \Lambda_1(\ell(t), m')) = \Lambda(t, \Lambda_2(\ell(t), m'))$$

and that Λ is injective it follows that $\Lambda_1(\ell(t), m') = \Lambda_2(\ell(t), m')$ for each $(t, m') \in F'_{\ell_1 \ell}$.

Furthermore if $(s, m') \in F'_{\ell_1} = F'_{\ell_2}$, there exists $t \in T$ such that $(\ell(t), m') \in F'_{\ell_1}$. Then $(t, m') \in F'_{\ell_1 \ell}$, thus $\Lambda_1(\ell(t), m') = \Lambda_2(\ell(t), m')$, therefore

$$\Lambda_1(s, m') = \Lambda_2(s, m').$$

This statement implies that $\Lambda_1 = \Lambda_2$. We conclude that (ℓ, Λ) is an epimorphism. □

This proposition has the following partial converse.

Proposition 3. *Let (E, p, T) and (F, q, S) be sheaves of sets with a full set of local sections. Suppose that $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ is an epimorphism. Then the function $\ell : T \to S$ is surjective.*

Proof. Let R be a topological space and $j_1,\ \ j_2 : S \to R$ be continuous functions such that $j_1 \ell = j_2 \ell$. Consider the sheaf of sets (R, id_R, R) where $id_R : R \to R$ is the identity function. Let σ be a global section for q. Define the functions $\Delta_1 : R_{j_1} \to F$ and $\Delta_2 : R_{j_2} \to F$ by $\Delta_1(s, r) = \sigma(s)$ and $\Delta_2(s, r) = \sigma(s)$. Then $(j_1, \Delta_1) : (F, q, S) \to (R, id_R, R)$ and $(j_2, \Delta_2) : (F, q, S) \to (R, id_R, R)$ are morphisms in $\mathfrak{S}$. Moreover if $(j_1 \ell, L_1) = (j_1, \Delta_1) \circ (\ell, \Lambda)$ and $(j_2 \ell, L_2) = (j_2, \Delta_2) \circ (\ell, \Lambda)$ then $L_1(t, r) = \Lambda(t, \Delta_1(\ell(t), r)) = \Lambda(t, \sigma(\ell(t)))$ for each $(t, r) \in R_{j_1 \ell}$ and $L_2(t, r) = \Lambda(t, \Delta_2(\ell(t), r)) = \Lambda(t, \sigma(\ell(t)))$ for each $(t, r) \in R_{j_2 \ell}$. Then $(j_1, \Delta_1) \circ (\ell, \Lambda) = (j_2, \Delta_2) \circ (\ell, \Lambda)$. On the other hand, since (ℓ, Λ) is an epimorphism, we have $(j_1, \Delta_1) = (j_2, \Delta_2)$, hence $j_1 = j_2$.
We have proved that ℓ is an epimorphism in the category of topological spaces and continuous functions. Then ℓ is surjective. □

The following example shows an epimorphism $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ such that Λ is not injective.

Example. Consider the Sierpinski space $T = \{0, 1\}$ whose open sets are $\emptyset$, T and $\{1\}$, $E_0 = \{a, b\}$, $E_1 = \{c\}$ and let E be the disjoint union of E_0 and E_1. Define $p : E \to T$ by $p(0, a) = p(0, b) = 0$ and $p(1, c) = 1$. Let $\sigma,\ \tau : T \to E$ be the global selections for p defined by $\sigma(0) = (0, a)$, $\sigma(1) = (1, c)$, $\tau(0) = (0, b)$ and $\tau(1) = (1, c)$. Existence Theorem guarantees that the ranges of the restrictions of σ and τ to open sets of T form a basis for a topology for E such that (E, p, T) is a sheaf of sets and σ and τ are sections. Now consider the sheaf (F, q, S) of sets where $S = \{0\}$, $F = \{a, b\}$ with the discrete topology and $q(m) = 0$ for $m = a,\ b$. Let $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ be the morphism defined by $\ell(t) = 0$ for each $t \in T$, $\Lambda(0, a) = (0, a)$, $\Lambda(0, b) = (0, b)$ and $\Lambda(1, a) = \Lambda(1, b) = (1, c)$. Now consider a sheaf (G, ρ, R) of sets and a pair (ℓ_1, Λ_1),

$(\ell_2, \Lambda_2) : (F, q, S) \to (G, \rho, R)$ of morphisms such that $(\ell_1, \Lambda_1) \circ (\ell, \Lambda) = (\ell_2, \Lambda_2) \circ (\ell, \Lambda)$. Since ℓ is surjective and $\ell_1 \ell = \ell_2, \ell$ we conclude that $\ell_1 = \ell_2$. Consider $(t, m) \in G_{\ell_1}$. From $(\ell_1, \Lambda_1) \circ (\ell, \Lambda) = (\ell_2, \Lambda_2) \circ (\ell, \Lambda)$ we have that $\Lambda(0, \Lambda_1(t, m)) = \Lambda(0, \Lambda_2(t, m))$, then either $\Lambda_1(t, m) = \Lambda_2(t, m) = a$ or $\Lambda_1(t, m) = \Lambda_2(t, m) = b$, therefore $\Lambda_1 = \Lambda_2$, thus (ℓ, Λ) is an epimorphism such that Λ is not injective.

5. Factorization of morphisms

Let (E, p, T) and (F, q, S) be sheaves of sets each one containing a full set of global sections, let

$$(\ell, \Lambda) : (E, p, T) \to (F, q, S)$$

be a morphism and consider the sub-space $\ell(T)$ of S.

Given $\alpha \in \Gamma(q)$ let α_Λ be the section for p defined by $\alpha_\Lambda(t) = \Lambda(t, \alpha(\ell(t)))$ for each $t \in T$ and let Σ be the set $\{\alpha_\Lambda : \alpha \in \Gamma(q)\}$. Denote by $(\widehat{E}, \widehat{p}, \ell(T))$ the sheaf of sets that is constructed by germination with change of the base space from the function p, the collection Σ and the co-restriction $\ell_1 : T \to \ell(T)$ of ℓ to $\ell(T)$.

Define $\Lambda_1 : \widehat{E}_{\ell_1} \to E$ by $\Lambda_1(t, [\alpha_\Lambda]_{\ell(t)}) = \alpha_\Lambda(t)$. Note that if $[\alpha_\Lambda]_{\ell(t)} = [\beta_\Lambda]_{\ell(t)}$ then there exist a neighborhood V of $\ell(t)$ such that $\alpha_\Lambda(s) = \beta_\Lambda(s)$ for every $s \in \ell^{-1}(V)$. Thus $\alpha_\Lambda(t) = \beta_\Lambda(t)$ and hence Λ_1 is well defined. Moreover Λ_1 is a continuous function because if $t_0 \in T$ and $\alpha \in \Gamma(q)$ then the range of $\alpha_\Lambda \restriction_W$, where W is a neighborhood of t_0, is a basic neighborhood of $\alpha_\Lambda(t_0)$. If $t \in W$ and if $[\beta_\Lambda]_{\ell(t)}$ is an element of the range of $\widehat{\alpha_\Lambda}) \restriction_W$ then $\widehat{\alpha_\Lambda}) \restriction_W (\ell(t)) = [\beta_\Lambda]_{\ell(t)}$, therefore $[\beta_\Lambda]_{\ell(t)} = [\alpha_\Lambda]_{\ell(t)}$. This implies that $\beta_\Lambda(t)$ is in the range of $\alpha_\Lambda \restriction_W$. Then Λ_1 is a continuous function.

In order to prove that (ℓ_1, Λ_1) is an epimorphism we will use the following result that has a generalized version in the context of bundles of uniform spaces [5].

Lema 1. *If γ_1 and γ_2 are two global sections for $\widehat{p}$ such that $\Lambda_1(t, \gamma_1(\ell(t))) = \Lambda_1(t, \gamma_2(\ell(t)))$ for each $t \in T$, then $\gamma_1 = \gamma_2$.*

Proof. Let $\gamma_1,\ \gamma_2 \in \Gamma(\widehat{p})$ and suppose that

$$\Lambda_1(t, \gamma_1(\ell(t))) = \Lambda_1(t, \gamma_2(\ell(t)))$$

for each $t \in T$. Consider the sections $(\gamma_1)_{\Lambda_1}$ and $(\gamma_2)_{\Lambda_1}$ for p defined from γ_1 and γ_2 by

$$(\gamma_1)_{\Lambda_1}(t) = \Lambda_1(t, \gamma_1(\ell(t)))$$

and $(\gamma_2)_{\Lambda_1}(t) = \Lambda_1(t, \gamma_2(\ell(t)))$, then $(\gamma_1)_{\Lambda_1} = (\gamma_2)_{\Lambda_1}$. Let

$$\ell(t) \in \ell(T)\ ,\ \gamma_1(\ell(t)) = [\alpha_\Lambda]_{\ell(t)}$$

and $\gamma_2(\ell(t)) = [\beta_\Lambda]_{\ell(t)}$. We claim that $[\alpha_\Lambda]_{\ell(t)} = [\beta_\Lambda]_{\ell(t)}$. Remark that $\widehat{\alpha_\Lambda}(\ell(t)) = \gamma_1(\ell(t))$ and $\widehat{\beta_\Lambda}(\ell(t)) = \gamma_2(\ell(t))$ therefore there exists a neighborhood V of $\ell(t)$ such that $\widehat{\alpha_\Lambda}(s) = \gamma_1(\ell(s))$ and $\widehat{\beta_\Lambda}(s) = \gamma_2(s)$ for each $s \in V$. Let $r \in \ell^{-1}(V)$. We have that

$$\begin{aligned}\alpha_\Lambda(r) &= \Lambda_1(r, [\alpha_\Lambda]_{\ell(r)}) = \Lambda_1(r, \widehat{\alpha_\Lambda}(\ell(r))) \\ &= \Lambda_1(r, \gamma_1(\ell(r))) = \Lambda_1(r, \gamma_2(\ell(r))) \\ &= \Lambda_1(r, \widehat{\beta_\Lambda}(\ell(r))) = \Lambda_1(r, [\beta_\Lambda]_{\ell(r)}) \\ &= \beta_\Lambda(r).\end{aligned}$$

It follows that $[\alpha_\Lambda]_{\ell(t)} = [\beta_\Lambda]_{\ell(t)}$. Then $\gamma_1(\ell(t)) = \gamma_2(\ell(t))$. We conclude that $\gamma_1 = \gamma_2$. ☐

Consider a sheaf of sets (G, ρ, R) and a pair of morphisms

$$(j_1, \Delta_1), (j_2, \Delta_2) : (\widehat{E}, \widehat{p}, \ell(T)) \to (G, \rho, R)$$

such that $(j_1, \Delta_1) \circ (\ell_1, \Lambda_1) = (j_2, \Delta_2) \circ (\ell_1, \Lambda_1)$. To prove that $(j_1, \Delta_1) = (j_2, \Delta_2)$, note that $j_1 \ell_1 = j_2 \ell_1$ implies $j_1 = j_2$ because ℓ_1 is surjective. On the other hand if $(\ell(t_0), a) \in G_{j_1}$ then $(t_0, a) \in G_{j_1 \ell_1}$. Let $\tau \in \Gamma(\rho)$ be a global section such that $\tau(j_1 \ell_1(t_0)) = a$. The functions $\gamma_1,\ \gamma_2 : \ell(T) \to \widehat{E}$ defined by

$$\gamma_1(\ell(t)) = \Delta_1(\ell(t), \tau(j_1 \ell_1(t)))$$

and

$$\gamma_2(\ell(t)) = \Delta_2(\ell(t), \tau(j_1 \ell_1(t)))$$

are sections for $\widehat{p}$ and $\Lambda_1(t, \gamma_1(\ell(t))) = \Lambda_1(t, \gamma_2(\ell(t)))$ because

$$\Lambda_1(t, \Delta_1(\ell(t), \tau(j_1 \ell_1(t)))) = \Lambda_1(t, \Delta_2(\ell(t), \tau(j_1 \ell_1(t))))$$

for each $t \in T$. Lema 1 allows us to conclude that $\gamma_1 = \gamma_2$. Then $\gamma_1(\ell(t_0)) = \gamma_2(\ell(t_0))$, thus $\Delta_1(\ell(t_0), a) = \Delta_2(\ell(t_0), a)$ hence $\Delta_1 = \Delta_2$. We have shown that (ℓ_1, Λ_1) is an epimorphism. ☐

It remains to be seen that there exists a monomorphism $(\ell_2, \Lambda_2) : (\widehat{E}, \widehat{p}, \ell(T)) \to (F, q, S)$ such that $(\ell_2, \Lambda_2) \circ (\ell_1, \Lambda_1) = (\ell, \Lambda)$.

Consider the inclusion function $\ell_2 : \ell(T) \to S$. For each $a \in F$ choose a section α^a for q such that $\alpha^a(q(a)) = a$. Define $\Lambda_2 : F_{\ell_2} \to \widehat{E}$ by $\Lambda_2(\ell(t), a) = [\alpha^a_\Lambda]_{\ell(t)}$. Suppose that $\alpha^a(q(a)) = \beta(q(a)) = a$. There exists a neighborhood V of $q(a) = \ell(t)$ such that $\alpha^a(s) = \beta(s)$ for each $s \in V$. Let $t' \in \ell^{-1}(V)$. We have that $\alpha^a_\Lambda(t') = \Lambda(t', \alpha^a(\ell(t'))) = \Lambda(t', \beta(\ell(t'))) = \beta_\Lambda(t')$. This implies that $[\alpha^a_\Lambda]_{\ell(t)} = [\beta_\Lambda]_{\ell(t)}$ and consequently Λ_2 is a well defined function. Now we prove that Λ_2 is a continuous function. Given $(\ell(t_0), a) \in F_{\ell_2}$, the range of $\widehat{\alpha^a_\Lambda} \restriction_U$ where U is an open neighborhood of $\ell(t_0)$ is a basic neighborhood of $[\alpha^a_\Lambda]_{\ell(t_0)}$ in E. Let $(\ell(t), b)$ be a pair such that $q(b) = \ell(t)$, $\ell(t) \in U$ and $b = \alpha^a(\ell(t))$. Let α^b a section for q such that $\alpha^b(q(b)) = b$. Since $\alpha^a(\ell(t)) = \alpha^b(\ell(t))$ there exists a neighborhood V of $\ell(t)$ such that $V \subset U$ and $\alpha^a(s) = \alpha^b(s)$ for every $s \in V$. If $r \in \ell^{-1}V$ then $\alpha^a_\Lambda(r) = \alpha^b_\Lambda(r)$. Thus $[\alpha^a_\Lambda]_{\ell(t_0)} = [\alpha^b_\Lambda]_{\ell(t_0)}$. We conclude that $\Lambda_2(\ell(t), b)$ is in the range of $\widehat{\alpha^a_\Lambda} \restriction_U$. Then Λ_2 is continuous.

We have that (ℓ_2, Λ_2) is a morphism in $\mathfrak{S}$ that is a monomorphism is due to the fact that ℓ_2 is an injection and Λ_2 is a surjection.

On the other hand, $\ell_2\ell_1 = \ell$ and if $\Delta : F_{\ell_2\ell_1} \to E$ is defined by $(t, a) = \Lambda_1(t, \Lambda_2(\ell_1(t), a))$ for each $(t, a) \in F_{\ell_2\ell_1}$ then

$$\begin{aligned}\Delta(t,a) &= \Lambda_1(t, [\alpha^a_\Lambda]_{\ell(t)}) = \alpha^a_\Lambda(t)\\ &= \Lambda(t, \alpha^a(\ell(t))) = \Lambda(t,a).\end{aligned}$$

We have proved that the morphism (ℓ, Λ) can be factored through an epimorphism and a monomorphism: $(\ell, \Lambda) = (\ell_2\ell_1, \Delta) = (\ell_2, \Lambda_2) \circ (\ell_1, \Lambda_1)$.

To apply the theory of **M. M. Clementino, E. Giuli** and **W. Tholen** [3], we ought to define classes $\mathcal{M}$ and $\mathcal{E}$ of monomorphisms and epimorphisms in a suitable manner.

Let $\mathcal{M}$ be the set of all the monomorphisms $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ of $\mathfrak{S}$ such that ℓ is an embedding (homeomorphism onto its image) and Λ is a surjection and let $\mathcal{E}$ be the set of all the epimorphisms $(\ell, \Lambda) : (E, p, T) \to (F, q, S)$ of $\mathfrak{S}$ such that, if $(a, b) \in F \bigvee F$ and $a \neq b$ then for each $V \in \mathcal{V}(q(a))$ there is $r \in \ell^{-1}(V)$ such that $\alpha^a_\Lambda(r) \neq \alpha^b_\Lambda(r)$ for some α^a and some α^b in $\Gamma(q)$ with $\alpha^a(\ell(t)) = a$ and $\alpha^b(\ell(t)) = b$.

Note that $(\ell_1, \Lambda_1) \in \mathcal{E}$ and $(\ell_2, \Lambda_2) \in \mathcal{M}$. In this case we say that the morphism (ℓ, Λ) has an $(\mathcal{E}, \mathcal{M})$-factorization.

Furthermore if (E, p, T), (F, q, S), (E^*, p^*, q^*) and (F^*, q^*, S^*) are sheaves of sets, if

$$(\ell_1, \Lambda_1) : (E, p, T) \to (F, q, S)$$

is in $\mathcal{E}$, if

$$({\ell^*}_2, \Lambda_2^*) : (E^*, p^*, T^*) \to (F^*, q^*, S^*)$$

is in $\mathcal{M}$ and if $({\ell^*}_1, \Lambda_1^*)$ and (ℓ_2, Λ_2) are morphisms such that the diagram

$$\begin{array}{ccc}(E,p,T) & \xrightarrow{(\ell_1,\Lambda_1)} & (F,q,S)\\ ({\ell^*}_1,\Lambda_1^*)\big\downarrow & & \big\downarrow(\ell_2,\Lambda_2)\\ (E^*,p^*,T^*) & \xrightarrow[({\ell^*}_2,\Lambda_2^*)]{} & (F^*,q^*,S^*)\end{array}$$

commutes, then there exists a unique morphism

$$(\ell^\flat, \Lambda^\flat) : (F, q, S) \to (E^*, p^*, T^*)$$

such that the diagram

$$\begin{array}{ccc}(E,p,T) & \xrightarrow{(\ell_1,\Lambda_1)} & (F,q,S)\\ ({\ell^*}_1,\Lambda_1^*)\big\downarrow & \swarrow(\ell^\flat,\Lambda^\flat) & \big\downarrow(\ell_2,\Lambda_2)\\ (E^*,p^*,T^*) & \xrightarrow[({\ell^*}_2,\Lambda_2^*)]{} & (F^*,q^*,S^*)\end{array}$$

commutes. Hence $\ell^\flat(s) = {\ell^*}_1(t)$, where t is an element of T such that $\ell_1(t) = s$ and $\Lambda^\flat : E^*_{\ell^\flat} \to F$ is defined by $\Lambda^\flat(s, m^*) = \Lambda_2(s, a^*)$ where $a^* \in F^*$ is such that $m^* = \Lambda_2^*(\ell^\flat(s), a^*)$ [5]. Note that ${\ell^*}_2(\ell^\flat(s)) = \ell_2(s) = q^*(a)$. The property above shows that the $(\mathcal{E}, \mathcal{M})$-factorizations are essentially unique [1]. Then the pair $(\mathcal{E}, \mathcal{M})$ is a proper system of factorization of morphisms in $\mathfrak{S}$ [1].

Definition. Consider a category $\mathcal{X}$ with a proper factorization system for morphisms $(\mathcal{E}, \mathcal{M})$. For every object X, the class sub(X) of all morphisms in $\mathcal{M}$ with codomain X is preordered by the relation $m \leq n$ if and only if there is a morphism j such that $n \circ j = m$. We write $m \cong n$ if $m \leq n$ and $n \leq m$. If $f : X \to A$ is a morphism and $m \in$ sub(X) then we denote the $\mathcal{M}$-part in the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$ by $f(m)$.

Definition. A *closure operator* c of $\mathcal{X}$ with respect to $\mathcal{M}$ [3] is given by a family of functions $c_X : \text{sub}(X) \to \text{sub}(X)$ $(X \in \mathcal{X})$ such that

(1) c is extensive ($m \leq c_X(m)$ for all $m \in$sub(X));

(2) c is monotone (If $m \leq n$ then $c_X(m) \leq c_X(n)$ for all m, $n \in$sub(X));

(3) every morphism $f : X \to Y$ is c-continuous, that is: $f(c_X(m)) \leq c_Y(f(m))$ for all $m \in$sub(X).

Consider again the Category $\mathfrak{S}$ of Sheaves of Sets. Take two sheaves of sets (E, p, T) and (F, q, S) and suppose that

$$(\ell, \Lambda) : (E, p, T) \to (F, q, S)$$

belongs to $\mathrm{sub}(F,q,S)$. Consider the sheaf $(\widehat{E},\widehat{p},\overline{\ell(T)})$ obtained by germination over $\overline{\ell(T)}$ from the function p, the set of all sections α_Λ such that α is a global section for q and the function ℓ, and consider the morphism

$$(\overline{\ell},\overline{\Lambda}) : (\widehat{E},\widehat{p},\overline{\ell(T)}) \to (F,q,S)$$

defined by $\overline{\ell}(s) = s$ and $\overline{\Lambda}(s,a) = [\alpha^a_\Lambda]_s$ for $\alpha^a \in \Gamma(q)$ such that $\alpha^a(q(a)) = a$. This is a monomorphism that belongs to $\mathrm{sub}(F,q,S)$.

Consider $c_{(F,q,S)} : \mathrm{sub}(F,q,S) \to \mathrm{sub}(F,q,S)$ given by $c_{(F,q,S)}(\ell,\Lambda) = (\overline{\ell},\overline{\Lambda})$. The family of functions $c_{(F,q,S)}$, where (F,q,S) is a sheaf of sets equipped with a full set of global sections, defines a closure operator of $\mathfrak{S}$ with respect to $\mathcal{M}$. Indeed:

(1) Let $(\ell,\Lambda) : (E,p,T) \to (F,q,S)$ be an monomorphism in $\mathrm{sub}(F,q,S)$. Then $c_{(F,q,S)}(\ell,\Lambda) = (\overline{\ell},\overline{\Lambda})$. Consider the morphism $(\widehat{\ell},\widehat{\Lambda}) : (E,p,T) \to (\widehat{E},\widehat{p},\overline{\ell(T)})$ defined by $\widehat{\ell}(t) = \ell(t)$ and $\widehat{\Lambda}(t,[\alpha_\Lambda]_{\ell(t)}) = \alpha_{\Lambda(t)}$. It follows that $(\overline{\ell},\overline{\Lambda}) \circ (\widehat{\ell},\widehat{\Lambda}) = (\ell,\Lambda)$. Hence $(\ell,\Lambda) \le (\overline{\ell},\overline{\Lambda}) = (\ell,\Lambda)$.

(2) Suppose that $(\ell_1,\Lambda_1) : (E_1,p_1,T_1) \to (F,q,S)$, $(\ell_2,\Lambda_2) : (E_2,p_2,T_2) \to (F,q,S)$ belong to $\mathrm{sub}(F,q,S)$ and $(\ell_1,\Lambda_1) \le (\ell_2,\Lambda_2)$. Let

$$(\ell,\Lambda) : (E_1,p_1,T_1) \to (E_2,p_2,T_2)$$

be a morphism such that $(\ell_2,\Lambda_2) \circ (\ell,\Lambda) = (\ell_1,\Lambda_1)$. Then $\ell_2\ell = \ell_1$ and therefore $\ell_1(T_1) = \ell_2(\ell(T_1))$ is a subset of $\ell_2(T_2)$. Define the morphism

$$(\overline{\ell},\overline{\Lambda}) : (\widehat{E_1},\widehat{p_1},\overline{\ell_1(T_1)}) \to (\widehat{E_2},\widehat{p_2},\overline{\ell_2(T_2)})$$

by $\overline{\ell}(s) = s$ and $\overline{\Lambda}(s,[\alpha_{\Lambda_2}]_s) = [\alpha_{\Lambda_1}]_s$. Note that if $\beta_{\Lambda_2} \in [\alpha_{\Lambda_2}]_s$ there is a neighborhood V of s such that $\beta_{\Lambda_2}(r) = \alpha_{\Lambda_2}(r)$ for each $r \in \ell_2^{-1}(V)$. In other words, $\Lambda_2(r,\beta(\ell_2(r))) = \Lambda_2(r,\alpha(\ell_2(r)))$ for each $r \in \ell_2^{-1}(V)$. Let $r \in \ell_1^{-1}(V)$, then $\ell_1(r) \in V$, thus $\ell_2(\ell(r)) \in V$, hence $\ell(r) \in \ell_2^{-1}(V)$, therefore

$$\Lambda_2(\ell(r),\beta(\ell_2(\ell(r))) = \Lambda_2(\ell(r),\alpha(\ell_2(\ell(r))).$$

This means that $\Lambda_2(\ell(r),\beta(\ell_1(r))) = \Lambda_2(\ell(r),\alpha(\ell_1(r)))$. We have that

$$\begin{aligned}\beta_{\Lambda_1}(r) &= \Lambda_1(r,\beta(\ell_1(r))) = \Lambda(r,\Lambda_2(\ell(r),\beta(\ell_1(r))) \\ &= \Lambda(r,\Lambda_2(\ell(r),\alpha(\ell_1(r))) = \Lambda_1(r,\alpha(\ell_1(r))) \\ &= \alpha_{\Lambda_1}(r).\end{aligned}$$

Then $[\beta_{\Lambda_1}]_s = [\alpha_{\Lambda_1}]_s$.

On the other hand a basic neighborhood of $[\alpha_{\Lambda_1}]_s$ contains the range of $\widehat{\alpha_{\Lambda_1}} \restriction_V$ where V is a neighborhood of s. If W is the range of $\widehat{\alpha_{\Lambda_2}} \restriction_V$ then $(V \times W) \cap \widehat{E}_{2\overline{\ell}}$ is a basic neighborhood of $(s,[\alpha_{\Lambda_2}]_s)$ and if $(r,[\alpha_{\Lambda_2}]_r) \in (V \times W) \cap \widehat{E}_{2\overline{\ell}}$ then $\overline{\Lambda}(r,[\alpha_{\Lambda_2}]_r) = [\alpha_{\Lambda_1}]_r$ is in the range of $\widehat{\alpha_{\Lambda_1}} \restriction_V$ so $\overline{\Lambda}$ is continuous.

Furthermore, $\overline{\ell_2}\overline{\ell} = \overline{\ell_1}$ and if $(s,a) \in F_{\overline{\ell_1}}$ then

$$\begin{aligned}\overline{\Lambda}(s,\overline{\Lambda_2}(\overline{\ell}(s),a)) = \overline{\Lambda}(s,\overline{\Lambda_2}(s,a)) &= \overline{\Lambda}(s,[\alpha^a_{\Lambda_2}]_s) \\ &= [\alpha^a_{\Lambda_1}]_s = \overline{\Lambda_1}(s,a)\end{aligned}$$

then $(\overline{\ell_2},\overline{\Lambda_2}) \circ (\overline{\ell},\overline{\Lambda}) = (\overline{\ell_1},\overline{\Lambda_1})$. We conclude that $(\overline{\ell_1},\overline{\Lambda_1}) \le (\overline{\ell_2},\overline{\Lambda_2})$.

(3) Let $(\delta,\Delta) : (F,q,S) \to (G,\rho,R)$ be a morphism and let $(\ell,\Lambda) : (E,p,T) \to (F,q,S)$ in $\mathrm{sub}(F,q,S)$. Consider the morphism

$$c_{(F,q,S)}(\ell,\Lambda) = (\overline{\ell},\overline{\Lambda}) : (\widehat{E},\widehat{p},\overline{\ell(T)}) \to (F,q,S)$$

where $(\widehat{E},\widehat{p},\overline{\ell(T)})$ is the sheaf constructed by germination over $\overline{\ell(T)}$ from the function p, the set of all sections α_Λ, where $\alpha \in \Gamma(q)$, and the function ℓ. Denote the elements of $\widehat{E}$ by $[\alpha_\Lambda]_s$, where $\alpha \in \Gamma(q)$ and $s \in \overline{\ell(T)}$. Let $(\delta,\Delta) \circ (\overline{\ell},\overline{\Lambda}) = (\delta\overline{\ell},\Phi) : (\widehat{E},\widehat{p},\overline{\ell(T)}) \to (G,\rho,R)$ then $\Phi : G_{\delta\overline{\ell}} \to \widehat{E}$ is defined by $\Phi(s,m) = \overline{\Lambda}(s,\Delta(\overline{\ell}(s),m))$. Now let $(\theta,\Theta) : (\widehat{E}^\dagger,\widehat{p}^\dagger,\delta(\overline{\ell(T)})) \to (G,\rho,R)$ be the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of $(\delta\overline{\ell},\Phi)$, where $(\widehat{E}^\dagger,\widehat{p}^\dagger,\delta(\overline{\ell(T)}))$ is the sheaf constructed by germination over $\delta(\overline{\ell}(\overline{\ell(T)})) = \delta(\overline{\ell(T)})$ from the function $\widehat{p}$, the set of all sections γ_Φ where $\gamma \in \Gamma(\rho)$ and the function $\delta\overline{\ell}$. Denote the elements of $\widehat{E}^\dagger$ by $[\gamma_\Phi]^\dagger_r$ where $\gamma \in \Gamma(\rho)$ and $r \in \delta(\overline{\ell(T)})$. Therefore $\theta : \delta(\overline{\ell(T)}) \to R$ is defined by $\theta(r) = r$ and $\Theta : G_\theta \to \widehat{E}^\dagger$ is defined by $\Theta(r,m) = [\gamma^m_\Phi]^\dagger_r$ where γ^m is a global section for ρ such that $\gamma^m(\rho(m)) = m$. Consider the morphism $(\delta,\Delta) \circ (\ell,\Lambda) = (\delta\ell,\Phi_1) : (E,p,T) \to (G,\rho,R)$ and let $(\theta_1,\Theta_1) : (E^§,p^§,\delta(\ell(T))) \to (G,\rho,R)$ be the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of $(\delta\ell,\Phi_1)$. Therefore $(E^§,p^§,\delta(\ell(T)))$ is the sheaf constructed by germination over $\delta(\ell(T))$ from the function p, the set of all sections γ_{Φ_1} where $\gamma \in \Gamma(\rho)$ and the function $\delta\ell$. Denote the elements of $E^§$ by $[\gamma_{\Phi_1}]^§_r$ where $\gamma \in \Gamma(\rho)$ and $r \in \delta(\ell(T))$. We have that $\theta_1 : \delta(\ell(T)) \to R$ is defined by $\theta_1(r) = r$ and $\Theta_1 : G_{\theta_1} \to E^§$ is defined by $\Theta_1(r,m) = [\gamma^m_{\Phi_1}]^§_r$ where $\gamma^m \in \Gamma(\rho)$ is a global section such that $\gamma^m(\rho(m)) = m$. Now consider the morphism

$$\begin{aligned}&c_{(G,\rho,R)}(\theta_1,\Theta_1) = \\ &\quad = (\overline{\theta_1},\overline{\Theta_1}) : (E^{§\maltese},p^{§\maltese},\overline{\delta\ell(T)}) \to (G,\rho,R)\end{aligned}$$

where $(E^{\S^{\maltese}}, p^{\S^{\maltese}}, \overline{\delta\ell(T)})$ is the sheaf constructed by germination over $\overline{\delta\ell(T)}$ from the function $p^{\S}$, the set of all sections γ_{Θ_1} where $\gamma \in \Gamma(\rho)$ and the function θ_1. Denote the elements of $E^{\S^{\maltese}}$ by $[\gamma_{\Theta_1}]_r^{\maltese}$ where $\gamma \in \Gamma(\rho)$ and $r \in \overline{\delta\ell(T)}$. Then $\overline{\theta_1} : \overline{\delta\ell(T)} \to R$ is defined by $\overline{\theta_1}(r) = r$ and $\overline{\Theta_1} : G_{\overline{\theta_1}} \to E^{\S^{\maltese}}$ is defined by $\overline{\Theta_1}(r, m) = [\gamma^m_{\overline{\Theta_1}}]_r^{\maltese}$ where $\gamma^m \in \Gamma(\rho)$ is such that $\gamma^m(\rho(m)) = m$.

In order to show that the morphism (δ, Δ) is c-continuous, we claim that exists a morphism

$$(\varphi, \Omega) : (\widehat{E}^{\dagger}, \widehat{p}^{\dagger}, \delta(\overline{\ell(T)})) \to (E^{\S^{\maltese}}, p^{\S^{\maltese}}, \overline{\delta\ell(T)})$$

such that $(\overline{\theta_1}, \overline{\Theta_1}) \circ (\varphi, \Omega) = (\theta, \Theta)$. This would prove that $(\theta, \Theta) \leq (\overline{\theta_1}, \overline{\Theta_1})$.

Define $\varphi : \delta(\overline{\ell(T)}) \to \overline{\delta\ell(T)}$ by $\varphi(r) = r$ and $\Omega : E^{\S}{}^{\maltese}_{\varphi} \to \widehat{E}^{\dagger}$ by $\Omega(r, [\gamma_{\Theta_1}]_r^{\maltese}) = [\gamma_{\Phi}]_r^{\dagger}$. To show that Ω is well defined, consider a global section μ for ρ such that $[\gamma_{\Theta_1}]_r^{\maltese} = [\mu_{\Theta_1}]_r^{\maltese}$. We claim that $[\gamma_{\Phi}]_r^{\dagger} = [\mu_{\Phi}]_r^{\dagger}$. There exists an open neighborhood W of r in $\overline{\delta(\ell(T))}$ such that $\gamma_{\Theta_1}(r') = \mu_{\Theta_1}(r')$ for each $r' \in \theta^{-1}(W) = W \cap \delta(\ell(T))$. Then $\Theta_1(r', \gamma(\theta_1(r'))) = \Theta_1(r', \mu(\theta_1(r')))$ for each $r' \in W \cap \delta(\ell(T))$. Therefore $[\gamma_{\Phi_1}]^{\S}_{r'} = [\mu_{\Phi_1}]^{\S}_{r'}$ for each $r' \in W \cap \delta(\ell(T))$.

Let V be an open neighborhood of r in R such that $V \cap \overline{\delta(\ell(T))} = W$ and take $s \in (\delta\overline{\ell})^{-1}(V)$. Then

$$\begin{aligned}\gamma_{\Phi}(s) &= \Phi(s, \gamma(\delta(\overline{\ell}(s)))) = \overline{\Lambda}(s, \Delta(\overline{\ell}(s), \gamma(\delta(\overline{\ell}(s)))))\\ &= \overline{\Lambda}(s, \gamma_{\Delta}(\overline{\ell}(s))) = [(\gamma_{\Delta})_{\Lambda}]_s\end{aligned}$$

and

$$\begin{aligned}\mu_{\Phi}(s) &= \Phi(s, \mu(\delta(\overline{\ell}(s)))) = \overline{\Lambda}(s, \Delta(\overline{\ell}(s), \mu(\delta(\overline{\ell}(s)))))\\ &= \overline{\Lambda}(s, \mu_{\Delta}(\overline{\ell}(s))) = [(\mu_{\Delta})_{\Lambda}]_s.\end{aligned}$$

To verify that $[(\gamma_{\Delta})_{\Lambda}]_s = [(\mu_{\Delta})_{\Lambda}]_s$ consider

$$t \in \ell^{-1}((\delta\overline{\ell})^{-1}(V)).$$

We have that $t \in \ell^{-1}((\overline{\ell})^{-1}(\delta^{-1}(V)))$, therefore

$$\delta(\ell(t)) \in V \cap \overline{\delta(\ell(T))} = W,$$

then $[\gamma_{\Phi_1}]^{\S}_{\delta(\ell(t))} = [\mu_{\Phi_1}]^{\S}_{\delta(\ell(t))}$. This implies $\gamma_{\Phi_1}(t) = \mu_{\Phi_1}(t)$, then

$$\Phi_1(t, \gamma(\delta(\ell(t)))) = \Phi_1(t, \mu(\delta(\ell(t)))),$$

therefore

$$\Lambda(t, \Delta(\ell(t), \gamma(\delta(\ell(t))))) = \Lambda(t, \Delta(\ell(t), \mu(\delta(\ell(t))))),$$

then

$$\Lambda(t, \gamma_{\Delta}(\ell(t))) = \Lambda(t, \mu_{\Delta}(\ell(t)))$$

and $(\gamma_{\Delta})_{\Lambda}(t) = (\mu_{\Delta})_{\Lambda}(t)$. This implies that $[(\gamma_{\Delta})_{\Lambda}]_s = [(\mu_{\Delta})_{\Lambda}]_s$ then $[\gamma_{\Phi}]_r^{\dagger} = [\mu_{\Phi}]_r^{\dagger}$. We conclude that Ω is well defined. The continuity of Ω is straighforward and the pair (φ, Ω) is a morphism in $\mathfrak{S}$ such that $(\overline{\theta_1}, \overline{\Theta_1}) \circ (\varphi, \Omega) = (\theta, \Theta)$. It follows that $(\theta, \Theta) \leq (\overline{\theta_1}, \overline{\Theta_1})$.

Definition. Let c be a closure operator of a category $\mathcal{X}$ with respect to $\mathcal{M}$. An object $A \in \mathcal{X}$ is *c-Hausdorff* if $u \circ c_X(m) = v \circ c_X(m)$ for all $u,\ v : X \to A$ and all $m \in \text{sub}(X)$ such that $u \circ m = v \circ m$.

Remark. In the Category of Topological Spaces and continuous functions with the usual closure c, the c-Hausdorff objects are the Hausdorff spaces. Consider the Category $\mathfrak{S}$ of Sheaves of Sets with the closure operator c defined above. If the sheaf of sets (E, p, T) is a c-Hausdorff object of $\mathfrak{S}$ then T is a Hausdorff space, indeed, if $u,\ v : X \to T$ are continuous functions and if $m : Z \to X$ is an embedding such that $um = vm$, then take the trivial sheaves of sets (X, id_X, X) and (Z, id_Z, Z) and the functions $\Lambda_1 : E_u \to X$,$\Lambda_2 : E_v \to X$ and $\Delta : M_m \to Z$ defined by $\Lambda_1(x, a) = x$, $\Lambda_2(x, a) = x$ and $\Delta(z, x) = z$. It is apparent that (u, Λ_1), $(v, \Lambda_2) : (X, id_X, X) \to (E, p, T)$ are morphisms of $\mathfrak{S}$, that

$$(m, \Delta) : (Z, id_Z, Z) \to (X, id_X, X)$$

belongs to $\text{sub}(X, id_X, X)$ and that $(u, \Lambda_1) \circ (m, \Delta) = (v, \Lambda_2) \circ (m, \Delta)$. Therefore $(u, \Lambda_1) \circ c_{(X, id_X, X)}(m, \Delta) = (v, \Lambda_2) \circ c_{(X, id_X, X)}(m, \Delta)$, then $u\overline{m} = v\overline{m}$, where $\overline{m}$ is the injection of $\overline{m}(Z)$ into X. Thus T is a Hausdorff space. $\square$

Definition. Let c be a closure operator of a category $\mathcal{X}$ with respect to $\mathcal{M}$. A morphism $f : X \to Y$ is *c-preserving* if $f(c_X(m)) \cong c_Y(f(m))$ for all $m \in \text{sub}(X)$.

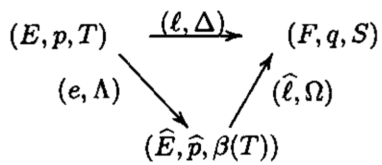
Definition. Let c be a closure operator of a category $\mathcal{X}$ with respect to $\mathcal{M}$. An object $X \in \mathcal{X}$ is *c-compact* if the product proyection $p_Y : X \times Y \to Y$ is c-preserving for every object $Y \in \mathcal{X}$.

Remark. In the Category $\mathfrak{S}$ of Sheaves of Sets with the closure operator c defined above, if the sheaf of sets (E, p, T) is a c-compact object of $\mathfrak{S}$ then T is a compact space in the Category of topological spaces and continuous function with the usual closure [5].

6. Stone-Čech compactification of a sheaf of sets

If F denotes the Forgetful Functor from the Category $\mathcal{C}$ of Compact Hausdorff Spaces (and continuous maps) to the Category $\mathcal{T}$ of Topological Spaces and if X denotes an arbitrary topological space, there exists a celebrated universal arrow $(e, \beta(X))$ from X to F known as the Stone-Čech compactification of X. To generalize this universal arrow we take in place of F the Forgetful Functor from the Category of Sheaves of Sets having Compact Hausdorff Base Space (and appropriately defined morphisms) to the Category of Sheaves of Sets and in place of X we take an arbitrary sheaf of sets.

Let (E,p,T) be a sheaf of sets with a full set of global sections. Let $e : T \to \beta(T)$ be the canonical function and Σ be the set of all the global sections for p. Let $(\widehat{E},\widehat{p},\beta(T))$ be the sheaf of sets obtained by germination from the function p, the family Σ and the function e. The sheaf $(\widehat{E},\widehat{p},\beta(T))$ is c-Hausdorff and c-compact. Furthermore if (F,q,S) is a c-Hausdorff and c-compact object of $\mathfrak{S}$ and if $(\ell,\Delta) : (E,p,T) \to (F,q,S)$ is a morphism, then there exists a unique morphism $(\widehat{\ell},\Omega) : (\widehat{E},\widehat{p},\beta(T)) \to (F,q,S)$ such that the diagram



commutes. Indeed, since (F,q,S) is c-Hausdorff and c-compact, the topological space S is Hausdorff and compact and since ℓ is continuous there exists a unique continuous function $\widehat{\ell} : \beta(T) \to S$ such that $\widehat{\ell}e = \ell$. Consider the set $F_{\widehat{\ell}} = \{(k,a) \in \beta(T) \times F : \widehat{\ell}(k) = q(a)\}$. For each $(k,a) \in F_{\widehat{\ell}}$ let α^a a global section for q such that $\alpha^a(q(a)) = a$ and let α^a_Δ the section for p defined by $\alpha^a_\Delta(t) = \Delta(t,\alpha^a(\ell(t)))$. Define $\Omega : F_{\widehat{\ell}} \to \widehat{E}$ by $\Omega(k,a) = [\alpha^a_\Delta]_k$. Consider $(k,a) \in F_{\widehat{\ell}}$ and let β^a be another global section for q such that $\beta^a(q(a)) = a$. We claim that $[\alpha^a_\Delta]_k = [\beta^a_\Delta]_k$. There exists a neighborhood V of $q(a)$ such that $\alpha^a(s) = \beta^a(s)$ for each $s \in V$ and $\widehat{\ell}^{-1}(V)$ is a neighborhood of k. Let $t \in e^{-1}(\widehat{\ell}^{-1}(V))$, then $t \in \ell^{-1}(V)$, thus $\ell(t) \in V$, hence $\alpha^a_\Delta(t) = \Delta(t,\alpha^a(\ell(t))) = \Delta(t,\beta^a(\ell(t))) = \beta^a_\Delta(t)$ and it follows that $[\alpha^a_\Delta]_k = [\beta^a_\Delta]_k$. On the other hand $\widehat{p}(\Omega(k,a)) = \widehat{p}([\alpha^a_\Delta]_k) = k$ for each $(k,a) \in F_{\widehat{\ell}}$. To see that $(\widehat{\ell},\Omega) \circ (e,\Lambda) = (\ell,\Delta)$, let $(\widehat{\ell}e,\Delta') = (\widehat{\ell},\Omega) \circ (e,\Lambda)$ where $\Delta' : F_{\widehat{\ell}e} \to E$ is defined by $\Delta'(t,a) = \Lambda(t,\Omega(e(t),a))$. We have

$$\begin{aligned}\Delta'(t,a) &= \Lambda(t,\Omega(e(t),a)) = \Lambda(t,[\alpha^a_\Delta]_{e(t)}) \\ &= \alpha^a_\Delta(t) = \Delta(t,\alpha^a(\ell(t))) \\ &= \Delta(t,\alpha^a(q(a))) = \Delta(t,a)\end{aligned}$$

and since $\widehat{\ell}e = \ell$ then $(\widehat{\ell},\Omega) \circ (e,\Lambda) = (\ell,\Delta)$. Now to establish the uniquess of $(\widehat{\ell},\Omega)$, suppose that $(\widehat{\ell}_1,\Omega_1) : (\widehat{E},\widehat{p},\beta(T)) \to (F,q,S)$ is a second morphism such that $(\widehat{\ell}_1,\Omega_1) \circ (e,\Lambda) = (\ell,\Delta)$. Then $\widehat{\ell}_1 = \widehat{\ell}$ because $\widehat{\ell}_1 e = \ell$. For $(k,a) \in F_{\widehat{\ell}}$, let τ be a global section for p (depending on k and a) such that $[\tau]_k = \Omega_1(k,a)$. Note that $\Omega_1(k,a) = \Omega_1(k,\alpha^a(\widehat{\ell}_1(k))) = \alpha^a_{\Omega_1}(k)$. Then $[\tau]_k = \alpha^a_{\Omega_1}(k)$. To show that $\Omega(k,a) = \Omega_1(k,a)$ it suffices to verify that $[\alpha^a_\Delta]_k = [\tau]_k$. If $t \in T$ then

$$\begin{aligned}\alpha^a_\Delta(t) &= \Delta(t,\alpha^a(\ell(t))) = \Lambda(t,\Omega_1(e(t),\alpha^a(\ell(t)))) \\ &= \Lambda(t,\Omega(e(t),\alpha^a(\ell(t)))),\end{aligned}$$

but

$$\begin{aligned}\Lambda(t,\Omega_1(e(t),\alpha^a(\ell(t)))) &= \Lambda(t,\Omega_1(e(t),\alpha^a(\widehat{\ell}_1(e(t))))) \\ &= \Lambda(t,\alpha^a_{\Omega_1}(e(t))) = (\alpha^a_{\Omega_1})_\Lambda(t)\end{aligned}$$

and

$$\begin{aligned}\Lambda(t,\Omega(e(t),\alpha^a(\ell(t)))) &= \Lambda(t,\Omega(e(t),\alpha^a(\widehat{\ell}(e(t))))) \\ &= \Lambda(t,\alpha^a_{\Omega}(e(t))) = (\alpha^a_{\Omega})_\Lambda(t),\end{aligned}$$

therefore $\alpha^a_\Delta(t) = (\alpha^a_{\Omega_1})_\Lambda(t) = (\alpha^a_\Omega)_\Lambda(t)$, then $\alpha^a_\Delta = (\alpha^a_{\Omega_1})_\Lambda = (\alpha^a_\Omega)_\Lambda$. We claim that $[(\alpha^a_{\Omega_1})_\Lambda]_k = [\tau]_k$. Since $[\tau]_k = \widehat{\tau}(k) = \alpha^a_{\Omega_1}(k)$, there exists a neighborhood V of k such that $\widehat{\tau}(r) = \alpha^a_{\Omega_1}(r)$ for each $r \in V$. If $r \in e^{-1}(V)$ then

$$\begin{aligned}(\alpha^a_{\Omega_1})_\Lambda(r) &= \Lambda(r,\alpha^a_{\Omega_1}(e(r))) = \Lambda(r,\widehat{\tau}(e(r))) \\ &= \Lambda(r,[\tau]_{e(r)}) = \tau(r).\end{aligned}$$

This implies that $[\tau]_k = [(\alpha^a_{\Omega_1})_\Lambda]_k$. Therefore $[\alpha^a_\Delta]_k = [\tau]_k$ and $\Omega(k,a) = \Omega_1(k,a)$.

We have proven the following statement:

Theorem (Stone-Čech Compactification). *If (E,p,T) is an object of $\mathfrak{S}$, then there exist a c-Hausdorff and c-compact object $(\widehat{E},\widehat{p},\beta(T))$ of $\mathfrak{S}$ and a morphism $(e,\Lambda) : (E,p,T) \to (\widehat{E},\widehat{p},\beta(T))$ in $\mathfrak{S}$ such that if (F,q,S) is a c-Hausdorff and c-compact object of $\mathfrak{S}$ and if $(\ell,\Delta) : (E,p,T) \to (F,q,S)$ is a morphism of $\mathfrak{S}$, then there exists a unique morphism $(\widehat{\ell},\Omega) : (\widehat{E},\widehat{p},\beta(T)) \to (F,q,S)$ of $\mathfrak{S}$ such that $(\widehat{\ell},\Omega) \circ (e,\Lambda) = (\ell,\Delta)$.*

The pair $((e,\Lambda),(\widehat{E},\widehat{p},\beta(T)))$ is called the Stone-Čech compactification of the sheaf of sets (E,p,T).

The first author has obtained a generalized version of the above development for the Category of Uniform Bundles [5].

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