KANT, FREGE, HILBERT AND THE SOURCES OF GEOMETRICAL KNOWLEDGE

by

Mario Laserna[1]

Resumen

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1.- La Revolución Científica del siglo XVII se fundamenta en la posibilidad de anticipar, a través de modelos matemáticos, las estructuras del mundo espacio temporal. De ahí surge la noción de ciencia experimental cuya finalidad está en determinar cuales de las posibles estructuras teóricas corresponden a los hechos espacio-temporales del mundo en que vivimos.

2.- Producida la Revolución Científica y destronado el psicologismo inherente al realismo aristotélico, la pregunta es: ¿Cómo son las estructuras de la mente que permiten la creación y desarrollo de la actividad científica lograda por la física matemática-experimental, y en particular, cual es la creación geométrica de la mente que sirve de base a la física moderna?

Ese es el problema sobre las fuentes del conocimiento geométrico que se plantea Kant, problema que es retomado por G. Frege, y posteriormente por David Hilbert.

3.- Todo ello lleva a una nueva concepción de la Ciencia, vista como una creación de la mente humana, para sistematizar su experiencia del mundo espacio-temporal habitado por *Homo sapiens*.

Palabras clave: Kant, Frege, Geometría mental, Hilbert, Historia.

Abstract

1.- The scientific revolution of the 18th century was based in capability of predicting, using mathematical models, the structure of the spacio-temporal world. From there, arose the notion of experimental science whose goals are determine wich of the theorical architectures correspond to spatio-temporal facts.

2.- Producing the scientific revolution and destroying the psychologism inherent in the realism of Arostotle is the question "What are the mental structures that permit the creation and development

1. Or a relation between objects?

of the scientific model achieved by mathematico-experimental physics? This is the problem concerning the sources of geometric knowledge proposed by Kant and later by Fregge and still later by David Hilbert.

3.- These lead to a new concept of science seen as creativity of the human mind in order to systematize our experience in the spatio-temporal world inhabited by *Homo sapiens*.

Key words: Kant, Frege, Mental Geometry, Hilbert, History.

I-Kant: between Frege and Hilbert?

Towards the end of his life Frege's search for a firm foundation for geometrical knowledge led him to conceptions akin to the kantian ones. This fact is the more remarkable in that kantian thought in this respect had fallen into complete disrepute in Frege's own days. The question, therefore, immediately arises whether there is more than a merely verbal approximation between kantian and fregean conceptions as to the meaning and validity of geometrical truths. Such a question may be readily answered not only by comparing what both these thinkers actually affirmed, but also, and most cogently, by comparing their assertions with the hilbertian attempts to base geometry on purely formal considerations, i.e., without raising the question as to the meaning of the geometrical entities nor as to the truth or falsify of the ensuing statements. In the course of our explorations into fregean and kantian theory so as to discover the sources of geometrical knowledge, it will become clear that the problem of non purely verbal coincidences must be stated at two different levels: a) Are they talking about the same problem: *do they agree on what it is that we mean by "straight line", "point", "plane", "intersection" and the other terms out of which geometrical statements are built*? b) Do they agree on the principles on which an answer to the questions posed by a) are to be given; in other words, do they share the criteria in terms of which the validity of an answer may be attained? Let us examine these two questions in detail.

Right from the beginning it may be argued that Kant and Frege agree in relation to a) on one basic issue: both accept that the problem of geometrical truth must be referred to an euclidean frame of reference. All serious kantian scholars know that Kant's attempt to explain the validity of euclidean truths is based on a traditional interpretation of Euclid. Kant did not re-interpret Euclid. He considered euclidean utterances as the data, the hard facts which demanded an explanation. Consequently, he invented a theory of the nature of geometrical propositions which should account as well for their a priori as for their synthetic character. Such characteristic notes of geometrical utterances mean *that they are not purely formal*, i.e., empty utterances, and second, that although they hold true of physical objects, they cannot be contradicted by empirical facts. These assertions are in flat contradiction to the hilbertian conception which would retort: In as much as geometrical truths are absolutely certain, they cannot apply to experience; i.e., are void of empirical reference. Conversely, if they possess empirical content they cannot be absolutely certain (a certainty which, according to Kant issues from their a priori character), in the einestenian formulation of the problem. (See Appendix A).

Now, with regard to Frege it is also clear that this "loyalty" to Euclid is a sine qua non requirement of the whole discussion. Only in terms of an euclidean conception of what an axiom is, can the problem be stated. If, in a hilbertian fashion, one is talking about objects and relations alien to the "euclidean "meanings" (whatever that may mean), one is solving the wrong problem. Which is precisely what Frege accuses Hilbert of doing: "When one looks at the words of one of his axioms (Hilbert's) it seems off hand that we are facing an axiom in the euclidean sense; but the simple words mislead, because every word is used differently than by Euclid". It is clear, consequently, that an euclidean universe of discourse constitutes the common ground on which Kant and Frege analyze the sources of geometrical knowledge, in sharp contrast to the formalism of the so-called "implicit definition" method of Hilbert. At this point we would like to make it clear, for reasons which should become apparent subsequently, that *the euclidean universe of discourse essential to the kantian analysis does not, in any way, restrict the validity of the kantian explanations to so called euclidean geometry. Kant's theory covers both euclidean and non-euclidean propositions*. In this respect he differs from Frege whose concept of why a geometrical axiom is valid is not only obscure, but narrow; in contrast to the impressive power and generality of the kantian notion of "concepts by construction".

We turn our attention now to the second question, namely, do Kant and Frege agree as to the criteria under

which a valid solution to the nature of geometrical truth may be given? In contrast to the first point raised, there exists no clear answer to this question. Nonetheless, it can be argued that certain fregean lines of thought point in the direction along which kantian thought, with its thoroughgoing analysis, found what is to be regarded as a highly coherent and satisfactory solution. However, the kantian answer is only attainable in a gradual way which demands, among other things, that one attains clarity with respect to certain fundamental concepts.

II-What is Space?

Strangely enough there is one basic kantian concept, the absence of which in Frege and Hilbert accounts for the fact that kantian thought agrees with Frege in some aspects –namely, the euclidean universe of discourse– and with Hilbert in others –namely, the free construction of geometrical concepts by the mind– in such a fashion that Kant may be considered the fortunate and coherent synthesis of two incomplete but complementary systems. *The kantian notion looming behind the fregean and hilbertian fragments is that of space.* For Kant space is not a concept abstracted from experience in the way "dog" or "golden mountain" are. Space is an entity that partakes of our sensorial nature in as much as all experience of objects presuppose space: "We represent to ourselves objects as out-side us, and all without exception in space... space is not an empirical concept which has been derived from outer experiences. For in order that certain sensations be referred to something outside me (That is, to something in another region of space from that in which I find myself)... the notion of space must be presupposed". B 37 -38.

It is clear, therefore, that space is intimately related to my experience of objects. If it were not for space, there would, in fact, exist no possibility of experiencing ordinary physical objects. In fact, Kant argues: "The notion of space, cannot, therefore, be obtained as an experience from the relations of outer appearance. On the contrary, such outer experience is itself possible only through the presence of such a notion. B 38 In consequence, space is a necessary notion prior to the experience of objects but intimately related to them since "we can never conceive to ourselves that space doesn't exist although one may well imagine that there be objects present in it". B 38

What must be retained from this kantian analysis of the nature of space is two fold: Firstly, I have the notion of space as something given prior to the experience of any concrete object; secondly, the nature of space is not intellectual. I do not "think" space. I rather "perceive" it in as much as it is the possibility *not of my thinking* but of my *perceiving* physical objects. It is, consequently, a notion closely related to physical objects not being itself a physical object. In kantian terms, it is a notion *of the possibility* of physical objects.

Which are the physical properties of objects most directly connected to space? Do there exist physical properties of objects which may be pre-figured, anticipated in space even when the physical object itself is not given? Certainly yes, answers Kant, and I believe no one will disagree with him: of all qualities of objects, *those only pertaining to form or shape can be anticipated:* "The shape of a cone we can form for ourselves in intuition; unassisted by any experience,... but the colour of this cone must be previously given in some experience or other". I: 743

It is this peculiarity of anticipating in space the geometrical properties expressed in form or shape what essentially *ties space to geometry.* Quite naturally, since *space is the possibility of physical objects* we may conclude that the science of space, namely, geometry, *is the science of the possibility of physical objects.* In other words, each time I anticipate a shape or form say a hexagon, I am anticipating a certain physical entity, namely, such a one as has this hexagonal shape. Geometry is consequently applicable to physical reality because it deals with a constitutive quality of all and each physical object, namely, to have shape or form. Why colour cannot be constructed a priori, is another question which does not impinge on the fact *that shape is a-priori.*

We conclude therefore, that geometry refers to the spatial qualities of objects and can therefore be developed independently from the factual, i.e., empirical existence of objects in as much as space is something given to our mind as a notion in which we can imagine, determine, and construct all sorts of shapes and forms[2]. We must keep clearly in mind, however, that space is a sensorial notion, something we "perceive" as creating the possibility of experiencing concrete physical objects.

It is not therefore a purely logical, formal notion as Hilbert claimed it to be. But neither is it physical reality. It is an intermediating entity existing prior to physical experience and making that experience possible.

Let us examine, now, how geometrical entities an the relations existing between them have spatial existence, attain truth in this specifically kantian sense. Spatial

2. And relations?

existence, i.e., shape, being independent or anticipatory of physical existence, has two prerequisites: *mind can create spatial objects at will provided it establishes the rule, operational definition, figurative schema, or principle of its creation in a way which is free from inherent or inner contradictions which hinder its construction*. Nothing, however, prevents me from constructing two figures "side by side" which are externally contradictory, say, a circle and a square. What I certainly am not free to construct is, say, a square-circle, for the simple reason that there is no guiding-principle or schema of its construction. Clearly, once an object is constructed it is "true" i.e., *exists as a spatial entity, no matter what the empirical physical world contains*. Constructivity implies possible experimentation, i.e., testability, empirical import.

Corresponding to my capacity of creative imagination we have, consequently, a whole array of spatial objects together with the schema, principle or operational rule of its construction. I have, therefore, as Kant says, constructed objects and their corresponding concept which, quite naturally, is also my own construction. A la Hilbert? That is the question! Yes, as far as my freedom of construction, but definitely not in as much as these constructed spatial concepts possess by their very origin a spatial interpretation namely, the constructed object! It is for this reason that they are concepts à la Frege or à la Euclid, i.e., they can claim to be apodictically true since they may never be contradicted by experience. Hilbert constructivism has no *built-in concrete interpretation because Hilbert, (and for that matter everyone else) never understood the kantian notion of space*. Geometrical truths are, consequently, those which can be exhibited, exemplified in a spatial construction, say, two points determine a straight line; or three points not on a straight line determine a triangle. This consideration leads us, indeed is identical to the second kantian condition imposed on concepts by construction, to wit: *Only those constructed concepts which are potentially exemplifiable in empirical experience are geometrically legitimate*. And conversely: Any physical object may be considered as to its shape or form as un example, a case, of a construable spatial form or shape. This second condition necessarily follows from the relation of space to physical objects as the condition of their possibility, i.e., *that without space no physical objects may be apprehended as existent*.

We may now pose the question: Once a spatial concept has been constructed, how do we know if a physical object falls under it, is subsumed, is an example of it? The obvious answer is: Through an experiment. The only way I can know if the table on which I am writing is square, round or ellipsoidal is by controlled observation. In other words, by asking nature, (in this case the spatial objects in my room) if the object on which I am writing falls under a certain concept the spatial definition or schema of which I am in full possession of. Such controlled observation, i.e., one based on a prior question, is what we term Experiment. It yields us knowledge through the fact that it answers a question:

Is my table round or square?

An experiment, therefore, is a method for establishing whether a given physical object falls under an operationally constructed concept! This is pure Kantian orthodoxy!

"Through the determination of pure intuition we can acquire a priori knowledge of objects, as in mathematics, but only in regard to their form as appearances; whether there can be things which must be intuited in this form, is still left undecided. Mathematical concepts are not, therefore, by themselves knowledge, except on the supposition that there are things which allow of being presented to us only in accordance with the form of that pure sensible intuition. Now things in space and time are given only in so far as they are perceptions (that is, representations accompanied by sensation) - therefore only through empirical representation". B 147

III-Sensorial transcendental idealism

We wish to emphasize the logical significance of the experimental method as a means of establishing whether a given space-time event (an object, a system of relations between objects) falls under a concept given prior to the experiment. It is not difficult to see that such a procedure constitutes the essence of the so called experimental method as employed by experimental physicists. It is not necessary, however, to restrict it to mathematical physics in spite of the fact that it is in this area of space-time events that it has acquired renown and admiration due to the mastery over nature it has given on man. *What is essential to the experimental method is the fact that it answers a question posed by the experimenter*. The peculiarity and importance of physics originates in the fact that it enables us to state questions which cannot be formulated in any other language! Obviously *this language is that proper to mathematics, as Galileo affirmed, or that of quantity, as Kant says*. But for the latter what is essential to mathematics is that it can construct its concepts *prior to any empirical apprehension of them*. It

follows that whatever type of construction of concepts which is intuitive and which can anticipate space-time event is to be considered mathematics. This Kant says quite explicitly and it is worthwhile quoting him: "Those who propose distinguish philosophy from mathematics by saying that the former has as its object QUALITY only and the latter QUALITY only, have mistaken the effect for the cause. The form of mathematical knowledge is the cause why it is limited exclusively to quantities. For it is the concept of quantities only that allows of being constructed" B742.

A few lines previously Kant had told us: "PHILOSOPHICAL Knowledge is the knowledge gained by reason from concepts; mathematical knowledge is the knowledge gained by reason from the construction of concepts". B741 Clearly, a concept like dog or green is philosophical since it is derived from experience and not constructed. It is readily seen that I am able to carry on an experiment if I say: "Is there a green dog in the kitchen?" and then proceed to inspect that space-time object called "my kitchen". Since I know what I am looking for, (a green dog) I am able to experience whether a certain object exists which falls under the concept "green dog in the kitchen". There is clearly no difficulty here since the concept I utilize for my experiment is derived from experience. In other words, it is a posteriori. *According to Kant the really important concepts for mastering nature do not originate in experience, i.e., do not acquire their meaning from ordinary sensorial experience in the way that I require a prior empirical input in order to think the concept.*

Concepts by construction exhibit the rare peculiarity that although they do not originate in experience nonetheless they apply to experience. Indeed, to such an extent that the whole of experimental physics depends on them! At this point does it not arouse the curiosity of the reader that all notions essential to developing modern experimental physics were formulated with such precision and insight by Kant, some hundred and twenty years before professional philosophers of science, not to mention the physicists themselves, even started wondering about them? If there exists a man ahead of his time in matters pertaining to philosophy of science and scientific method it certainly was the secluded and punctilious old Prussian of Scottish descent!

The question which has now become central is: What are concepts by construction? How is it that although given prior to experience (otherwise we could not construct them) they apply to experience? For it is clear that *they cannot be, like hilbertian constructions, void of sensorial meaning, consequently totally disconnected from experience.* How can a concept be totally a priori, indeed my own invention, and still be related to a reality which I do not invent? Which is given objectively, as something real? Such is the great question to which Kant must supply an answer if he is not to be considered as the confused old gentleman most philosophers of science consider him to be. Clearly nothing can be so meaningless, or even downright erroneous and preposterous as to provide an answer to a problem which "has not been dreamt of" in the philosophy of those who are to be your judges.

And on to the kantian problem of concepts by construction! "To construct a concept means to exhibit a priori the intuition which corresponds to the concept", *It is now vital to understand the most misunderstood of all kantian key terms, namely, "intuition".* Most people take "intuition" to mean some mysterious non-rational fashion of reaching a truth, a conclusion about something. Mothers have intuitions as to why their babies don't want to eat; police inspectors as to who is the true murderer and mathematicians as to which proposition is true, before you have carried on the demonstration. *None of this has anything to do with Kant's intuition.*

In his use of the term, which corresponds to the latin one, *intuition is the direct apprehension of an individual object or event. It is the immediate way of experiencing something THROUGH sense affectation.* For the human mind, intuition has an essential sensorial nature. Its most frequent, albeit natural way, is through empirical sensation, say seeing or touching, smelling an apple. Here are some examples of Kant's use of "intuition": "...in being affected by objects, it (the subject) obtains *immediate representation,* that is intuition of them..."B41 "...intuition takes place only in so far as the object is given to us..."B33 or also "since no representation, save when it is an intuition, is in immediate relation to an object..." B93. It happens also, and this is of the greatest importance, that *the human mind can only have such immediate representations* (we would also say "immediate experience") *of objects through sensorial data.* In other words, to *intuit an object is to have an immediate sensorial affectation by it.* This Kant repeats abundantly and little doubt should be left in the reader as to the essentially sensorial nature of an intuition. "*Our mode of intuition is dependent on the existence of the object...*" B72 It is therefore not a grasping of truth or knowledge in the void but in something very concrete, and, moreover, something apprehended not by any mysterious parapsychological faculty, but by something common to all men: our sensorial faculties. ...*Our kind of intuition does not extend to all things, but only to objects of our senses.*" B342; and also: "without sensibility we cannot have any

intuition..." B92 We must conclude, therefore, that "*intuitive evidence*" or "*intuitively acquired meaning*" *pertains to evidence or meaning acquired through sense apprehension*, an essentially sound, interpersonal and repeteable procedure, as good a guarantee of solid knowledge as any we can think of. Such a position implies:

"Now, as the Aesthetic has shown, the only intuition possible to us is sensible; consequently, the thought of an object in general, by means of a pure concept of understanding, can become knowledge for us only in so far as the concept is related to objects of the senses. Sensible intuition is either pure intuition (space and time) or empirical intuition of that which is immediately represented, through sensation, as actual in space and time". B 146-147

And now Kant draws a conclusion that should rejoice the most radical operationalist or logical positivist: "We cannot, therefore, positively extend the sphere of the objects of our thought beyond the conditions of our sensibility" B343 Such thinking would be meaningless, void of any content.

And now comes the remarkable kantian discovery: *Although all intuition for human minds is sensorial, not all intuition is empirical! What at first sight appears as a contradiction is, in reality, the key to the whole kantian doctrine of concepts by construction.* We can have, thanks to space and time, non-empirical sensorial intuitions! *In other words, we are able to have sensorial meaning and evidence without the customary perceptive experience*! Such is what Kant calls a pure intuition; or an a priori intuition; withstanding its pure or a priori character it remains sensorially conditioned. *It is NON-INTELLECTUAL. And to have grasped this fact (essentially belonging to the euclidean universe of discourse) is considered by Kant one of his major intelectual achievements.* The doctrine itself is clearly stated in numerous occasions in the Critique of Pure Reason, for example in B 150:"But since there lies in us a certain form of a priori sensible intuition...; and in B151: "This synthesis of the manifold of sensible intuition, which is possible and necessary a priori, may be entitled *figurative* synthesis..." It turns out that this figurative synthesis is nothing less than the capacity of constructing objects in space. It actually involves an act of imagination whereby "Imagination is the faculty of representing in intuition (i.e., in a sensorially operational meaningful way) an object that is not itself present, (not present physically, but schematically anticipated, naturally)".

And the thought continues: "Now since all our intuition is sensible, the imagination, owing to the subjective condition under which alone intuit gives to the concepts of understanding a corresponding intuition, belongs to sensibility". B 151

We shall presently expound is what implied by this relation between concepts of understanding "and the corresponding intuition" which must necessarily be sensible, although not tied with actual perception. For the moment we wish to quote Kant in one of the rare moments in which he indulges in self praise. Quite naturally, in a matter that had escaped the inquisitive eye of all previous thinkers although the problem, the universal validity of geometrical propositions, was considered beyond dispute.

Idealism proper always has a visionary purpose, and can have no other, but my idealism is solely for conceiving the possibility of our knowledge a priori of the objects of experience, which is a problem that has not hitherto been solved, or even raised. There by falls the whole visionary idealism, which always (as can heady be seen from Plato) inferred from our knowledge a priori (even that of geometry) another intuition than that of the senses (namely intellectual intuition), because it never occurred to anyone that the senses should intuit a priori.

It is clear, by now, that Kant's so called transcendental idealism should be called "transcendental sensorial idealism" blocking thereby all the misconceptions and misinterpretations that have arisen out of the term "intuition", interpreted in a non-kantian fashion!

As far as modern philosophical vocabulary goes, one could well say "schematic construction with empirical reference", instead of "pure intuition" or "intuition a priori"; a fairly simple change of vocabulary which dissipates a lot of terminological darkness and exhibits the kantian doctrine in all its modernity and lucidity.

IV-Space as a neutral substratum

Keeping in mind that space is also an intuition, i.e., something immediately given to our sensorial faculties, not in experience but as the condition of all possible experience of physical objects, we are in a position to clearly understand the meaning of: "*To construct a concept means to exhibit a priori the intuition which corresponds to the concept. For the construction of a concept, we therefore need a non-empirical intuition. The latter must, as intuition, be a single object.*" B741

If we paraphrase in modern vocabulary we have: "*To construct a concept means to provide the operational*

definition which fits any object which may fall under the concept. The operational definition must offer the possibility of identifying individual objects as falling under the concept." In other words, in carrying out an experiment we always know how an object, eventually given, must behave if it is to fall under the concept. *It is such a construction of objects in pure intuition, as anticipations of empirical intuition which guarantee the truth of geometrical axioms. Once the a priori object is constructed, no empirical object can refute its validity simply because any experiment can only show if a given empirical case falls under a concept or not.* It can never show that the a priori constructed concept is spatially non-existent since *the spatial construction constitutes its very existence*. We shall consider this in more detail in the next section when discussing Frege's position vis à vis the parallel axiom. For the present we wish to establish that according to Kant, space is not tied up with any kind of geometry, euclidean or non-euclidean. Space is undetermined as far as the objects that may be constructed in it.

Here is Kant's statement:

"Space is something so homogeneous and in respect of all particular properties so indeterminate that there is certainly no hoard of natural laws to be found in it. On the other hand, that which determines space for understanding, is that it contains the ground of the unity of the construction of these figures. The mere universal form of intuition that is called space is indeed the substratum of all intuitions which can be correlated to particular objects, and admittedly there lies in space the condition of the possibility and variety of intuitions. But the unity of objects is determined solely by the understanding, and according to conditions which lie in the nature of the understanding" Prolegomena 38.

Space is neutral. It is neither euclidean nor non-euclidean since both classes of objects are construable a priori. Otherwise one could not be carrying out experiments to find out whether certain PHYSICAL intuitions are euclidean or not. This truth is however, very difficult to grasp by persons who do not distinguish between space and physical reality; or between physical reality and its possibility, namely, space.

The well established fact of modern theoretical systematisation, namely, that wider theories exhibit their validity by deductively including narrower ones, (relativity theory in relation to newtonian mechanics) fits very well under the kantian scheme of concepts by construction. One can consider the facts of our newtonian universe as one highly complex space - time event and the numerous experiments which verify newtonian mechanics as a huge one single additive experiment which shows how the newtonian fact falls under the einstenian constructed concept. Quite obviously the personal psychological purposes and motivations of the experimenters – whether to perform just a fragment of the all-embracing newtonian-einstenian experiment, is totally irrelevant. That the one all-embracing experiment took one hour or one century to be performed in no way affects the basic conceptual pattern.

V- The Frege - Hilbert Debate.

It will be our task now to show how Hilbert, in his attempts to obtain a secure foundation for geometry also, without ever realizing it, arrived at certain concepts which are central to the kantian theory of geometrical propositions. The reader quite naturally intuits that we are referring to the by now familiar concepts by construction. *The fact that neither Frege nor Hilbert ever attained the kantian conception of space prevented their developing of a coherent conception of euclidean geometry as a series of propositions which possess both an a priori and a synthetic nature.* We have made it clear that space as a general substratum open to any concept which can be operationally defined in it allows both euclidean and non-euclidean concepts.

A geometrical axiom does not refer to any concrete physical object. What it does is to establish the conditions a physical object must satisfy to fall or be subsumed under it. It is consequently clear that any set of axioms which is non-contradictory defines a geometrical spatial system. Whether our real existent objects satisfy one or the other system is a matter for physical experimentation. Such is the kantian doctrine of geometry and I believe no one can quarrel with it.

Frege made the mistake, in his rather vehement dispute with Hilbert, to place the truth of geometrical axioms in an impossible, ill-defined category. He insistently asserted that axioms are meaningful statements and that, in the euclidean sense, they are absolute truths. *However, not having at his disposal the kantian notion of space, he maintained that if an axiom is true, then what denies it must be false.* Consequently, if the parallel axiom is true, then the riemanian alternative must be false. *Such an argument would be valid if the truth of the fifth euclidean postulate were based on relations existing between physical objects AND, moreover, the riemanian alternative were referred to the SAME physical objects.*

But we know that this is not, and as far as Euclid is concerned, never was, the euclidean sense of truth in regard to geometrical propositions, since *concepts by construction determine the concrete real objects which make them true in physics a posteriori*, i.e., by means of an experiment that reveals that the HERE AND NOW, concrete objects satisfy the axioms. This was seen partially by Frege. In the articles of 1906 "On the Foundations of Geometry" published in the Annual Report of the Union of German Mathematicians he polemizes with Korselt, a full fledged hilbertian, and says: "I demand from a definition of a point, that according to it one can decide whether a given object, say my pocket watch, is a point. Korselt misunderstands me as saying the question should be decidable purely from the definition without any assistance from perception, and affirms that this is not possible. Completely right! Whether a given stone is a diamond cannot be decided only by the definition of the word "diamond". But one can demand from the explanation given, that it decide the question objectively in such a way that anyone who has knowledge of the stone in question can decide whether it is a diamond". If to this fregean position one were to add the notion of concepts by construction as inherent to our notion of space, Frege would have solved the puzzle of how euclidean axioms are true. Clearly "diamond" or any non-constructed concept can never lead to an adequate answer because of their origin in empirical, non-a priori reality. Kant argues:

"The a priori method gives us a rational and mathematical knowledge through the construction of the concept; the a posteriori method our merely empirical (mechanical) knowledge which is incapable of yielding necessary and apodictic propositions. Thus I might analyse my empirical concept of gold without gaining anything more than merely an enumeration of everything that I actually think in using the word, thus improving the logical character of my knowledge but not in any way adding to it. But I take the material body, familiarly known by this name, and obtain perceptions by means of it; and these perceptions yield various propositions which are synthetical but empirical. When the concept is mathematical as in the concept of a triangle, I am in a position to construct the concept, that is, to give it a priori in intuition, and in this way to obtain knowledge which is at once synthetic and rational" B749.

The applicability of mathematics to nature is based on such constructibility of its concepts: "Through the determinations of pure intuition we can acquire a priori knowledge of objects, as in mathematics but only in regard to their form, as appearances; whether there can be things which must be intuited in this form, is still left undecided. Mathematical concepts are not, therefore, by themselves knowledge, except on the supposition that there are things which allow of being presented to us only in accordance with the form of that pure sensible intuition". B 147. Which constructed concepts correspond to actual physical objects, located, as corresponds to their very nature, in space, we determine by means of an experiment.

It is now interesting to see how if, under the kantian guidance, one starts with euclidean concepts by construction, one can end up caught in the logical meshes of an Hilbertean system. *It suffices to "ascend" one degree in the activity of concept construction.* We thus abandon spatially defined schemas and arrive at pure logical forms.(In other words, the propositions of the system become more abstract; Nevertheless, they maintain their structural or formal type). The procedure is the same one used by Kant in the process of constructing a concept on the basis of a pure intuition, i.e., of apprehending a concrete object: "...the method of geometrical construction, by means of which I combine in a pure intuition (just as I do in empirical intuition) the manifold which belongs to the schema of a triangle in general and therefore to its concept" B746.

The step-by-step method may be stated thus: Given certain physical objects we focus on their quality "shape". In a second stage, as in the case of the ancient Greeks, "a new light flashes upon the mind" since we find that we can consider any particular shape in separation from the object it adheres to. We arrive thus at different shapes, say circular, rectangular and so on. It is with the next step'that geometry is born since: "The true method, so he found (Thales or some other) was not to inspect what he discerned either in the figure, or in the bare concept of it". B XII.

The human mind has now reached spatial constructions, the properties of which are expressed (it could not be otherwise) in synthetic a priori judgements. It is now clear that a further step can be taken: Since these synthetic a priori judgements possess a logical structure, it is possible to consider such a logical structure in isolation from the spatially constructed concepts. In order to obtain this, it suffices to "drop" the spatially meaningful concepts. This can be done either by leaving blank spaces, in the place they occupy, thereby transforming say, "two points determine a line" into "two... a...". Or by simply ignoring what the words "point" and "line" mean, this second procedure being the one employed by Hilbert and so vehemently criticized by Frege. The result is in both cases what we shall call an axiom form, which in turn

becomes an axiom once we fill in or substitute appropriate geometrical terms or entities in the blank spaces. The characteristics of an axiom form are:

a) *They are not propositions either of the a priori type like those of geometry, or of the empirical type of physics.*

b) *They may become propositions either of geometry or of physics according to what one substitutes in the blank spaces.*

The question now is whether the substitution of geometrical entities is equivalent to the substitution of physical objects, since in both cases we obtain statements capable of being either true or false. Clearly, it is the nature of this truth or falsity which differs accordingly if we fill in geometrical, i.e., concepts by construction, or physical objects. In the first case we can only obtain true statements and never a false one because of the very nature of the operation we are performing, namely, constructing a concept. In the second, since in reality we are carrying on an experiment which presupposes the concept under which certain concrete objects may be subsumed or not, the ensuing statement may be either true or false.

The reason for this essential logical difference in the nature of the two statements originates in the fact that physical objects are something given whereas constructed concepts. Have the properties I ascribe to them. And the act of filling in or substituting the blank spaces with geometrical terms is precisely intuiting, indeed, in a pure fashion, the spatial object out of which ensues the concept and the judgement. In order for such a filled-in axiom-form to yield a false statement, it would be necessary that the entities utilized to fill in the blank spaces should possess previously given characteristics, a condition which, as we have shown, contradicts their very nature of being concepts by construction.

In this respect Hilbert was totally justified in saying that the very fact of their construction guaranteed the truth of his axioms. *The error originated in not being able to define how those filled-in objects were constructed. He, Hilbert, attributed to the blank spaces a purely logical frame of reference which permitted that one fill in any kind of objects.* Which is what Frege did with sarcastic and undisguised delight when he argued: "When one finds that these axioms, (the hilbertian ones) do not express a proposition that, (according to, Korselt), should not worry us. Such linguistic expressions do not pretend to be descriptions of known facts. They only open the possibility of them. And truly with great subtlety: for example, that every sausage has two ends, or that any child may swing at least two small flags. Such (the formal axiom void of meaning) is evidently a description of a fashion in which objects of experience –the flags– can be associated with one another. The domain of applicability of this axiom is exactly as extended as objects of experience may be substituted in it".

Notwithstanding Frege's sharp analysis of the Hilbert attempts we reach the conclusion that his evaluation of the Hilbert method is incorrect. What Hilbert attempts is, in kantian terms, to attain concepts by construction. But since he lacks the concept of space as an a priori sensible intuition, he, Hilbert, finds no way of rendering his terms spatially and thereby experimentally meaningful. Hence, any physical object, including those lacking operational spatial significance, say sausages, pocket-watches or children waving at least two small flags in a patriotic parade, may be filled in. *A substitution which as such is not absurd. It becomes so, only when the axiom forms are considered as originally intended to describe spatial phenomena.*

VI-The non-inductive character of experimental physics.

Once the role played in experimental physics by concepts by construction is grasped the error of considering physics as an inductive science is easily seen. *Inductive procedures are only possible when objects are identified by means of a substance-attribute method.* It is then possible to ask: Are all lions yellow? Observation may exhibit, say, a blue lion. However, if the concept lion were constructed a priori, all its characteristics would be fully included in it since, as Kant says: "*mathematics is the only science that has definitions. For the object which it thinks It exhibits a priori in intuition, and this object certainly cannot* count *either more or less than the concept, since it is through the definition that the concept* of the object is given, and given originally, that is, without its being necessary to derive the definition from any other source". B757-758.

Clearly, therefore, there is not possibility of inductive confirmation or falsifiability. However, if I select a class of objects which fall, or so I believe, under a certain constructed concept, say, the chemical element carbon, C12, it may well occur that some concrete objects, i.e., samples, are not experimentally subsumed under C12. What do I infer? That my concept C12 is mistaken? What could that mean? *What modern chemistry concludes is certainly not that my concept of C12 is false, but that I*

made an error of judgement in believing the given samples would fall under C12. However great progress is made if I construct the new concept C14 and discover that my "rebellious" samples fall under it.

It is clear now that the falsifying experiment is an invitation to construct new concepts under which a rebellious portion of reality can be subsumed. Therefore it becomes most interesting to invent all sorts of concepts so as to bring as much space-time reality under them as possible. That is precisely the task of mathematical physics and its experimental branch. *It is non-inductive; it is a single experiment science and it is open to new discoveries of the type constructed - concept and corresponding experiment, but it is certainly non-inductive.* Sciences like zoology or astronomy, which deal with empirical concepts, i.e., such in which there exists or permits an essence-accident classification, are indeed inductive.

Conclusions

We have seen that Kant's theory of space leaves open the possibility that different geometrical systems, euclidean and non-euclidean, may be considered as true spatially, *in as much as each of them can be experimentally tested.* If certain physical objects Mi satisfy the axioms E, there is no reason why another group of objects Mj cannot satisfy the set of axioms R which would mean that E and R are not only spatially, but also empirically true. However, this does not even remotely mean that E and R are, taken together, non-contradictory. *In this regard Frege is wrong because he takes the meaning of each axiom to be independent of the meaning of the others, whereas in Hilbert thought it is the whole system taken in its totality which gives meaning to the parts or elements.* This is essential to concepts by construction. *Moreover, in the kantian case, once the axioms are constructed, i.e., shown to be consistent (otherwise we could not construct them), the objects necessarily exist; not as a result of their logical consistency but, here is the kantian "touch", because of their constructibility in that pure intuition we call space.* Clearly such guarants that the object Ezts does not apply to the example Hilbert gives in the next sentences of his letter, since arithmetical *entities are not constructed freely*: "The proposition 'each equation has a root' is true, or the existence of the root is proven as soon as the axiom 'each equation has a root' may be added to the other axioms of arithmetic without ever it being the case that a contradiction may arise".

Within the kantian theory Frege and Hilbert are both right and wrong. The reason being that the kantian point of view is more embracing and permits, even more, demands a redefinition of key concepts like *axiom, model, object, consistency,* truth, etc. It seems to me, thereby, proven that the higher point of view which Hilbert sought in order to consider the different geometries as subordinate to a more general point of view which would permit an analysis of the basic logical problems of geometry (truth, independence, consistency, etc.) is provided by Kant's theory of space and of concepts by construction. An accomplishment which bee is to fullfill the dream of the young man who at twenty three years of age" reflected on" a science of all possible types of space which would be the highest geometry that a finite mind could conceive".

The quotations from Kant¨s Critique of Pure Reason are taken from the standard german text KdrV published by Verlag von Felix Meiner in Hambur; the English translations are done by the author of the present Essay, An customary in Kantian literature, a numeral

Followed by a capital B corresponds to the page of the Second edition of the -KdrV given in Meiner.

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APPENDIX A

Carta de A. Einstein a M. Laserna (Septiembre, 1953).

"The institute for advanced study Princeton, New Jersey

September 22, 1953

Dr. Mario Laserna
Universidad de Los Andes
Calle 18-A Carrera 1-E
Apartado Aéreo 1976
Bogotá, Colombia

Dear Dr. Laserna:

I gather from your dialogue that a considerable difference of opinion exists between us with regard to this problem. Because I am *not* of the opinion that there exists an essential difference between concepts and methods in the fields of "common sense" and science.

Every linguistic utterance is wholly confined to the conceptual sphere. Concepts, as far as they have any basis, are –judged logically– free inventions of the mind (together with propositions connecting them). But those concepts and propositions receive their value and justification exclusively through their only intuitively given connection with perceptions (Erlebnissen). There is no logical way to deduce concepts and propositions from our crude experiences ("induction"). This is equally true for concepts like "red", "tree", as for concepts like "distance", atom", etc. The difference lies in the fact that scientific concepts and propositions are mostly brought into connection with sense-perceptions in a more indirect and complicated way. Also the use of numbers doesnot involve a difference in essence between scientific and common sense methods.

Apart from these differences of opinion I must confess that I do not want to appear in this field as a responsible partner. I have not studied epistomology thouroughly enough and I am not sufficiently aquainted with the tremendously extented literature in that field. It is enough if you mention that we had several discussions about these questions.

With kind regards,
Albert Einstein"

APPENDIX B

Carta de A. Einstein a M. Laserna (Enero, 1955)

January 8, 1955

"Dr. Mario Laserna
Universidad de Los Andes
Bogotá, Colombia

Dear Dr. Laserna,

Thank you for letter of December 14th.

I still believe that one cannot distinguish, in principle, between primary and secondary qualities. It is basic for all physics that one assumes a real world existing independently from any act of perception. But this we do not *know*. We take it only as a programme in our scientific endeavours. This programme is, of course, pre-scientific and our ordinary language in already based on it.

The concepto body-object and shape are not given to us directly by our sense-impressions but are a result of a mental construct. That this is not so easy to see is only produced by the fact that those steps made by every one of us in early childhood seem to us logically necessary. But this is not so.

With Kind regards,
sincerely yours,
Albert Einstein"

The quotations from Kant's Critique of Pure Reason are taken from the standard german text KdrV published by Verlag von Felix Meiner in Hambur; the English translations are done by the author of the present Essay, An customary in Kantian literature, a numeral

Followed by a capital B corresponds to the page of the Second edition of the -KdrV given in Meiner.[1] Carrera 3ª No. 9-52, Bogotá D.C. Colombia, Fax (057 1) 2848921.

1. i.e. are operationally meaningful.