ON THE QUANTUM STRUCTURE OF THE UNIVERSAL ENVELOPING ALGEBRA OF THE LIE ALGEBRA $\mathcal{ST}(2)$

by

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Abstract

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The structure of Hopf co-Poisson algebra on the universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$ of Lie algebra $\mathcal{ST}(2)$ is determined with the help of a solution of the Yang–Baxter equation. Using this solution, a bracket on the dual space of Lie algebra $ST(2)$ is also determined. This cobracket on $ST(2)$ induces a deformation of the universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$ which has a Hopf algebra structure, as we shall verify. This Hopf algebra is called the quantum group associated to a universal enveloping algebra.

Key words: Lie bialgebras, Hopf algebras, Poisson brackets, Lie Poisson group, Hopf co-Poisson algebra, Universal enveloping algebra, $r-$matrix, Quantum group, Yang–Baxter equation.

Resumen

Con ayuda de una solución de la ecuación clásica de Yang–Baxter determinamos la estructura de álgebra de Hopf-co-Poisson del álgebra envolvente universal $\mathcal{U}(\mathcal{ST}(2))$ del álgebra de Lie $\mathcal{ST}(2)$. Usando esta solución determinamos un corchete en el espacio dual del álgebra de Lie. Este co-corchete sobre $\mathcal{ST}(2)$ induce una deformación del álgebra envolvente universal $\mathcal{U}((\mathcal{ST}(2))$ que tiene estructura de álgebra de Hopf, como probaremos. Esta álgebra de Hopf es llamada el grupo cuántico asociado a un álgebra envolvente universal.

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Palabras claves: Biálgebra de Lie, Algebra de Hopf, Grupo de Lie Poisson, co-corchete, Algebra de Hopf co-Poisson, Algebra Envolvente Universal, $r-$matriz, ecuación de Yang Baxter. Grupo Cuántico.

1. Introduction

Let G be a Lie Group and $\mathcal{G}$ its Lie algebra; we can obtain quantum groups as deformations of the algebra of C^∞ functions $\mathcal{F}(G)$ on G, or as quantizations of a Lie bialgebra $\mathcal{G}$. A quantization of a Lie bialgebra $\mathcal{G}$ is a deformation of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ equipped with the co-Poisson Hopf algebra structure, such that the classical limit of this quantization is the Lie bialgebra structure of $\mathcal{G}$. To construct a deformation of the universal enveloping algebra we need to describe the co-Poisson Hopf algebra structure on $\mathcal{U}(\mathcal{G})$ or, equivalently, we must build the bialgebra structure on the Lie algebra $\mathcal{G}$.

The purpose of this work is to describe a mathematical procedure to produce a quantum group structure associated to a universal enveloping algebra, the universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$ of Lie algebra $\mathcal{ST}(2)$. To achieve this purpose we use a solution of the classical Yang–Baxter equation (CYBE) on Lie algebra $\mathcal{ST}(2)$. We build a cobracket in $\mathcal{ST}(2)$ connected with this solution. This cobracket determines a bialgebra structure in $\mathcal{ST}(2)$ and the co-Poisson-Hopf algebra structure in the universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$ as we shall verify.

We will deform the comultiplication in $\mathcal{U}(\mathcal{ST}(2))$ by means of a parameter h in order to build the algebra $\mathcal{U}_h(ST(2))$. We shall find appropriate expressions for the coproduct Δ_h, the antipode application S_h and the bracket $[\ ]_h$ on $\mathcal{U}_h(ST(2))$. We shall prove that $\mathcal{U}_h(ST(2))$, with these applications, has a Hopf algebra structure so that when $h \longrightarrow 0$ the coalgebra structure of $\mathcal{U}_h(ST(2))$ coincides with the bialgebra structure of $\mathcal{ST}(2)$. That is, we shall prove that this algebra is a quantum group of the universal enveloping algebra type.

2. The group $ST(2)$ and its algebra $\mathcal{U}(\mathcal{ST}(2))$

2.1. The Lie algebra $\mathcal{ST}(2)$

Let $ST(2)$ be the Lie group of upper triangular matrices 2×2 with determinant equal to 1 such that the operation of the group is the multiplication between matrices.

The Lie algebra $\mathcal{ST}(2)$ associated to $ST(2)$ is the Lie algebra of upper triangular matrices 2×2 with null trace on $\mathbb{R}$, where the matrices

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.1}$$

form a basis with the Lie bracket given by

$$[X_1, X_2] = -[X_2, X_1] = 2X_2,$$
$$[X_i, X_i] = 0,\ i = 1, 2 \tag{2.2}$$

2.2. The Lie bialgebra structure on $\mathcal{ST}(2)$

A Lie bialgebra is a Lie algebra with a Lie co-algebra structure δ fulfilling the 1-cocycle condition (2.5) with respect to the tensorial adjoint representation, (see [6] p. 43).

The Lie bialgebra structures may be induced by the adjoint aplication of Lie algebras

$$\mathrm{ad} : \mathcal{ST}(2) \rightarrow \mathcal{ST}(2) \times \mathcal{ST}(2)$$

defined by

$$\mathrm{ad}_X(Y) = [X, Y], \text{ for } X, Y \in \mathcal{ST}(2).$$

The adjoint representation of $\mathcal{ST}(2)$ is totally determined by its representation on the $\{X_1, X_2\}$ basis of Lie algebra, given by

$$\mathrm{ad}_{X_1}(X_1) = [X_1, X_1] = 0$$
$$\mathrm{ad}_{X_1}(X_2) = [X_1, X_2] = 2X_2$$
$$\mathrm{ad}_{X_2}(X_1) = [X_2, X_1] = -2X_2$$
$$\mathrm{ad}_{X_2}(X_2) = 0.$$

Any representation of a Lie algebra can be extended to a unique representation on the tensorial product of Lie algebras. Then we can extend the adjoint representation just defined to the adjoint tensorial representation in the following way

$$(\mathrm{ad}_X \otimes I + I \otimes \mathrm{ad}_X) : \mathcal{ST}(2) \otimes \mathcal{ST}(2) \rightarrow \mathcal{ST}(2) \otimes \mathcal{ST}(2)$$

$$\begin{aligned}(\mathrm{ad}_X \otimes I + I \otimes \mathrm{ad}_X)(X_i \otimes X_j) &= (\mathrm{ad}_X \otimes I + I \otimes \mathrm{ad}_X)(X_i \otimes X_j) \\ &= (\mathrm{ad}_X(X_i) \otimes IX_j + IX_i \otimes \mathrm{ad}_X(X_j)) \\ &= [X, X_i] \otimes X_j + X_i \otimes [X, X_j].\end{aligned}$$

With this adjoint tensorial representation and with one special r-tensor, $r \in \mathcal{ST}(2) \otimes \mathcal{ST}(2)$, we are able to define a co-Lie algebra structure.

An element $r \in \mathcal{ST}(2) \otimes \mathcal{ST}(2)$ defines a Lie bialgebra structure if and only if r is skew-symmetric and $[[r,r]] = 0$. The equation $[[r,r]] = 0$ is called the clasical Yang-Baxter equation (CYBE). Moreover if r is a skew-symmetric element, r is called an r-matrix.

We know that the Lie algebra $\mathcal{ST}(2)$ has only one $r-$matrix (see [9]), and that this r-matrix is the tensor

$$r = X_1 \otimes X_2 - X_2 \otimes X_1, \tag{2.3}$$

X_1, X_2 being the basis elements of $\mathcal{ST}(2)$. Thus r is skew-symmetric and satisfies the Yang Baxter equation,

$$[[r,r]] = [r_{12} + r_{13}] + [r_{12} + r_{23}] + [r_{13} + r_{23}] = 0.$$

where

$$r_{12} = X_1 \otimes X_2 \otimes I - X_2 \otimes X_1 \otimes I$$
$$r_{23} = I \otimes X_1 \otimes X_2 - I \otimes X_2 \otimes X_1$$
$$r_{13} = X_1 \otimes I \otimes X_2 - X_2 \otimes I \otimes X_1$$

Proposition 2.1. *The r-matrix*

$$r = X_1 \otimes X_2 - X_2 \otimes X_1$$

induces a cobracket on the Lie algebra $\mathcal{ST}(2)$ by the application

$$\delta : \mathcal{ST}(2) \longrightarrow \mathcal{ST}(2) \otimes \mathcal{ST}(2)$$

defined by

$$\delta(X) = (\mathrm{ad}_X \otimes I + I \otimes \mathrm{ad}_X)(r) = X \cdot r,$$

for $X \in \mathcal{ST}(2)$.

Proof. We can show that δ satisfies the properties of a cobracket. For this purpose it is enough to prove that δ satisfies these properties on $\{X_1, X_2\}$, the basis of the Lie algebra. In those elements, δ is given by

$$\begin{aligned}\delta(X_1) &= X_1.r = (\mathrm{ad}_{X_1} \otimes I + I \otimes \mathrm{ad}_{X_1})(X_1 \otimes X_2 - X_2 \otimes X_1)\\ &= -[X_1, X_2] \otimes X_1 + X_1 \otimes [X_1, X_2]\\ &= 2(X_1 \otimes X_2 - X_2 \otimes X_1),\\ \delta(X_2) &= X_2.r = (\mathrm{ad}_{X_2} \otimes I + I \otimes \mathrm{ad}_{X_2})(X_1 \otimes X_2 - X_2 \otimes X_1)\\ &= [X_2, X_1] \otimes X_2 - X_2 \otimes [X_2, X_1] = 0.\end{aligned} \tag{2.4}$$

Then δ satisfies the following cobracket properties over $\{X_1, X_2\}$:

(1) If $\delta(X) = \sum X_i \otimes X_j$ then

$$X_i \otimes X_j = -X_j \otimes X_i \quad \text{for all} \quad i, j.$$

(2) δ satisfies the associative property,

$$(\mathrm{id} \otimes \delta) \circ \delta - (\delta \otimes \mathrm{id}) \circ \delta = 0.$$

(3) δ is one 1-cocycle, that is, δ satisfies

$$\begin{aligned}&\delta([X_i, X_j])\\ &= (\mathrm{ad}_{X_i} \otimes I + I \otimes \mathrm{ad}_{X_i})\delta(X_j) - (\mathrm{ad}_{X_j} \otimes I + I \otimes \mathrm{ad}_{X_j})\delta(X_i)\\ &= X_i.\delta(X_j) - X_j.\delta(X_i)\end{aligned} \tag{2.5}$$

with $i, j = 1, 2$ and $X_i, X_j \in \mathcal{ST}(2)$.

The proof of these properties is straightforward. However, since δ is defined by an r-matrix we can deduce, from Proposition 2.1.2 of [**2**], that δ is a cobracket on $(\mathcal{ST}(2))$. Therefore $(\mathcal{ST}(2), \delta)$ is a Lie bialgebra.

The Lie bialgebra $(\mathcal{ST}(2), \delta)$ is called a quasitriangular bialgebra because it is generated by a solution of the CYBE and triangular bialgebra because it arises from a skew-symetric solution of the CYBE. This Lie bialgebra is also called a coboundary Lie bialgebra because the cobracket δ is a 1-cocycle.

2.3. The universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$

Let $\mathcal{ST}(2)$ be the Lie algebra of $ST(2)$ and let $\mathcal{T}(\mathcal{ST}(2))$ be the tensorial algebra of $\mathcal{ST}(2)$,

$$\mathcal{T}(\mathcal{ST}(2)) = \oplus_{n \geq 0} \mathcal{T}^n(\mathcal{ST}(2)) = \oplus_{n \geq 0} (\mathcal{ST}(2))^{\otimes^n}$$

where

$$\mathcal{T}^0 \mathcal{ST}(2) = (\mathcal{ST}(2))^0 = k, \quad \mathcal{T}^1 \mathcal{ST}(2) = (\mathcal{ST}(2)),$$

$$\mathcal{T}^n(\mathcal{ST}(2)) = (\mathcal{ST}(2)) \otimes (\mathcal{ST}(2)) \otimes \cdots \otimes (\mathcal{ST}(2)).$$

The universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$ of the Lie algebra $\mathcal{ST}(2)$ is the associative algebra,

$$\mathcal{U}(\mathcal{ST}(2)) \cong \mathcal{T}(\mathcal{ST}(2))/\mathcal{I},$$

where $\mathcal{I}$ is the ideal of $\mathcal{T}(\mathcal{ST}(2))$ engendered by

$$X_1 \otimes X_2 - X_2 \otimes X_1 - 2X_2, \qquad X_1, X_2 \in \mathcal{ST}(2).$$

with the product given by the recurrent operation,

$$\begin{aligned}(X_1^m X_2^n)(X_1^p X_2^q) &= X_1^{m-1} X_1 X_2 X_2^{n-1} X_1^p X_2^q\\ &= X_1^{m-1}(2X_2 + X_2 X_1) X_2^{n-1} X_1^p X_2^q\\ &= 2X_1^{m-1} X_2^n X_1^p X_2^q + X_1^{m-1} X_2 X_1 X_2^{n-1} X_1^p X_2^q\\ &= \cdots\end{aligned}$$

Note 1. It is known that a universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ is a Hopf algebra (see [**2**],

[11], [15], [16]) with the linear applications Δ, ε and S defined on the basis of $\mathcal{G}$ by

$$\begin{aligned} \Delta(X_i) &= X_i \otimes I + I \otimes X_i \qquad i = 1,2,\dots \\ \varepsilon(X_i) &= 0\ , \quad i = 1,2,\dots \\ S(I) &= I\ , \quad S(X_i) = -X_i\ , \quad i = 1,2\ . \end{aligned}$$

Then $\mathcal{U}(\mathcal{ST}(2))$ is a Hopf algebra with the operations just defined.

Note 2. It is known that a cocycle on $\mathcal{G}$ induces a co-Poisson structure in the universal enveloping algebra $\mathcal{U}(\mathcal{G})$, (see [3], [4], [6], and proposition 6.2.3 in [2]). Then the cobracket induced by δ in (2.4) satisfies the following cobracket properties on $\mathcal{U}(\mathcal{ST}(2))$

(1) Compatibility between δ and Δ,

$$\delta(X_iX_j) = \delta(X_i)\Delta(X_j) + \Delta(X_i)\delta(X_j).$$

(2) Co-Jacobi identity, that is δ satisfies the co-chain

$$\mathcal{G} \xrightarrow{\delta} \mathcal{G}\otimes\mathcal{G} \xrightarrow{\delta\otimes\mathrm{id}} \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G} \xrightarrow{\Sigma} \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}$$

where Σ means the sum over permutations of the factors in the triple tensor product.

(3) Co-Leibniz identity,

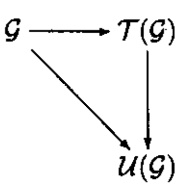
$$(\Delta\otimes id)\delta = (\mathrm{id}\otimes\delta)\Delta + \sigma_{23}(\delta\otimes\mathrm{id})\Delta$$

where σ_{23} means the permutations of the last two elements in $\mathcal{ST}(2)\otimes\mathcal{ST}(2)\otimes\mathcal{ST}(2)$.

Thus, from Note 1 and Note 2, we can infer that the universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$ of the Lie algebra $\mathcal{ST}(2)$ is a Hopf co-Poisson algebra with the coproduct Δ, the counit ε, the antipode S and the cobracket δ. These applications are defined on the generators X_1, X_2 by

$$\begin{aligned} \Delta(X_i) &= X_i\otimes I + I\otimes X_i\ , \quad i = 1,2\ ; \\ \varepsilon(X_i) &= 0, \quad i = 1,2\ ; \\ S(X_i) &= -X_i\ , \quad i = 1,2\ ; \\ \delta(X_1) &= 2(X_1\otimes X_2 - X_2\otimes X_1) \\ \delta(X_2) &= 0\ , \end{aligned} \tag{2.6}$$

which are extended to the elements of the algebra $\mathcal{U}(\mathcal{ST}(2))$ by the following commutative diagram



2.4. The Quantum algebra $\mathcal{U}_h(\mathcal{ST}(2))$

The quantum enveloping algebra $\mathcal{U}_h(\mathcal{ST}(2))$ of the Lie bialgebra $(\mathcal{ST}(2))$ is a quantization of $\mathcal{U}(\mathcal{ST}(2))$ when $\mathcal{U}(\mathcal{ST}(2))$ is considered as a co-Poisson-Hopf algebra. It means that $(\mathcal{ST}(2))$ has a quasitriangular structure(see [4]). Since the bialgebra $(\mathcal{ST}(2))$ has a quasitriangular structure we need work with exponential functions, it means that we should work over the ring $\mathbb{R}[[h]]$ of the formal series in h.

In order to build a quantization of $\mathcal{U}(\mathcal{ST}(2))$ we consider the Lie algebra $\mathcal{U}(\mathcal{ST}(2))[[h]] = \mathcal{U}_h(\mathcal{ST}(2))$ of formal power series in $\mathbb{R}[[h]]$ with coefficients in $\mathcal{U}(\mathcal{ST}(2))$, generated by X_1, X_2, I with the defining relation

$$[X_1, X_2]_h = 2\,\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}}$$

and

$$[X_i, X_i]_h = 0\ , \qquad \text{for } \ i = 1,2.$$

We must prove that $\mathcal{U}(\mathcal{ST}(2))[[h]]$ with this new product, has a Hopf $*$-algebra structure, so that when $h \longrightarrow 0$ the coalgebra structure of $\mathcal{U}_h(ST(2))$ coincides with the bialgebra structure of $\mathcal{ST}(2)$.

We shall find appropriate expressions for the coproduct Δ_h, the counit ε_h and the antipode application S_h defined on $\mathcal{U}(\mathcal{ST}(2))[[h]]$. For this purpose we take Δ_h as the deformation of co-proctuct Δ defined on $\mathcal{U}(\mathcal{ST}(2))$. Let be Δ_h the linear aplication

$$\Delta_h : \mathcal{U}_h(ST(2)) \longrightarrow \mathcal{U}_h(ST(2))\otimes\mathcal{U}_h(ST(2))$$

defined by

$$\Delta_h = \Delta + \frac{h}{2}\delta + O(h^2) \tag{2.7}$$

such that $\lim_{h\longrightarrow 0}\Delta_h \longrightarrow \Delta$. Here δ is the cobracket (2.4) and h is the parameter of the deformation. The map Δ_h in X_1, X_2 is given by

$$\begin{aligned} \Delta_h(X_1) &= X_1\otimes(I + hX_2) + (I - hX_2)\otimes X_1 \\ \Delta_h(X_2) &= X_2\otimes I + I\otimes X_2. \end{aligned} \tag{2.8}$$

Since $(I + hX_2)$ and $(I - hX_2)$ are functions in $\mathcal{U}(\mathcal{ST}(2))[[h]]$ we can write more generally,

$$\begin{aligned} \Delta_h(X_1) &= X_1\otimes f(h) + g(h)\otimes X_1 \\ \Delta_h(X_2) &= X_2\otimes f(h) + g(h)\otimes X_2. \end{aligned} \tag{2.9}$$

and since we must have $\lim_{h\longrightarrow 0}\Delta_h = \Delta$, f and g must satisfy

$$\lim_{h\longrightarrow 0} f(h) = I\ , \qquad \lim_{h\longrightarrow 0} g(h) = I$$

For these two functions we have:

Lemma 2.1. *For any choice of f and g, the application*

$$\Delta_h : \mathcal{U}_h(ST(2)) \to \mathcal{U}_h(ST(2)) \otimes \mathcal{U}_h(ST(2))$$

defined by (2.9) is an homomorphism of Lie algebras.

Proof. In fact Δ_h satisfies

$$\Delta_h([X_1, X_2]) = [\Delta_h(X_1), \Delta_h(X_2)] \tag{2.10}$$

whereas

$$\begin{aligned}
[\Delta_h(X_1), \Delta_h(X_2)] &= \Delta_h(X_1)\Delta_h(X_2) - \Delta_h(X_2)\Delta_h(X_1) \\
&= (X_1 \otimes f(h) + g(h) \otimes X_1)(X_2 \otimes f(h) + g(h) \otimes X_2) - (X_2 \otimes f(h) + g(h) \otimes X_2)(X_1 \otimes f(h) + g(h) \otimes X_1) \\
&= (X_1X_2 \otimes f(h) + g(h) \otimes X_1X_2 + X_1 \otimes f(h)X_2 + g(h)X_2 \otimes X_1) \\
&- (X_2X_1 \otimes f(h) + g(h) \otimes X_2X_1 + X_2g(h) \otimes X_1 + X_1 \otimes X_2f(h)) \\
&= (X_1X_2 - X_2X_1) \otimes f(h) + g(h) \otimes (X_1X_2 - X_2X_1) \\
&= [X_1, X_2] \otimes f(h) + g(h) \otimes [X_1, X_2] = \Delta_h([X_1, X_2]).
\end{aligned}$$

Lemma 2.2. *The application Δ_h defined by (2.9) is coassociative if and only if f and g satisfy*

$$\Delta_h(f) = f \otimes f \qquad \Delta_h(g) = g \otimes g$$

Proof. We must proof that Δ_h satisfies

$$(\Delta_h \otimes \mathrm{id})\Delta_h(X) = (\mathrm{id} \otimes \Delta_h)\Delta_h(X) \tag{2.11}$$

if and only if, f and g satisfy

$$\Delta_h(f) = f \otimes f \ , \ \Delta_h(g) = g \otimes g \tag{2.12}$$

In fact, the right-hand side of (2.11) takes the form

$$\begin{aligned}
(\Delta_h \otimes \mathrm{id})\Delta_h(X_i) &= (\Delta_h \otimes \mathrm{id})(X_i \otimes f(h) + g(h) \otimes X_i) \\
&= \Delta_h(X_i) \otimes f(h) + \Delta_h(g(h)) \otimes X_i \\
&= X_i \otimes f(h) \otimes f(h) + g(h) \otimes X_i \otimes f(h) \\
&\qquad + \Delta_h(g(h)) \otimes X_i \qquad (i = 1, 2)
\end{aligned}$$

and the left-hand side of (2.11) takes the form

$$\begin{aligned}
(\mathrm{id} \otimes \Delta_h)\Delta_h(X_i) &= (\mathrm{id} \otimes \Delta_h)(X_i \otimes f(h) + g(h) \otimes X_i) \\
&= X_i \otimes \Delta_h(f(h)) + g(h) \otimes \Delta_h(X_i) \\
&= X_i \otimes \Delta_h(f(h)) + g(h) \otimes (X_i \otimes f(h) + g(h) \otimes X_i) \\
&= X_i \otimes \Delta_h(f(h)) + g(h) \otimes X_i \otimes f(h) \\
&\qquad + g(h) \otimes g(h) \otimes X_i
\end{aligned}$$

We see that this two expressions are equal if, and only if,

$$\Delta_h(f(h)) = f(h) \otimes f(h) \text{ and } \Delta_h(g(h)) = g(h) \otimes g(h)$$

If we take the application

$$\varepsilon_h : \mathcal{U}_h(ST(2)) \to \mathbb{R} \ ,$$

defined by

$$\varepsilon_h(X_1) = \varepsilon_h(X_2) = 0 \ , \tag{2.13}$$

then ε_h is a counit for the application Δ_h in (2.9), that is, Δ_h and ε_h satisfy

$$\Delta_h(\mathrm{id} \otimes \varepsilon) = \Delta_h(\varepsilon \otimes \mathrm{id})$$

trivially. We can infer from Lemmas (2.1) and (2.2) that the map Δ_h in (2.9), with ε_h in (2.13), is a co-product on the space $\mathcal{U}_h(ST(2))$. Besides, if we consider the map δ in (2.4) extended on $\mathcal{U}_h(ST(2))$ we have the following assertion:

Proposition 2.2. *The space $\mathcal{U}_h(\mathcal{ST}(2))$ generate by X_1, X_2 is a Lie bialgebra, with the coproduct Δ_h the counit ε_h and the cobracket δ define by*

$$\begin{aligned}
\Delta_h(X_1) &= X_1 \otimes f(h) + g(h) \otimes X_1 \\
\Delta_h(X_2) &= X_2 \otimes f(h) + g(h) \otimes X_2 \\
\varepsilon_h(X_1) &= \varepsilon_h(X_2) = 0 \\
\delta(X_1) &= 2(X_1 \otimes X_2 - X_2 \otimes X_1) \\
\delta(X_2) &= 0
\end{aligned} \tag{2.14}$$

where f and g are functions that satisfy the following properties:

$$\lim_{h \to 0} f(h) = I \ , \quad \lim_{h \to 0} g(h) = I$$

$$\Delta_h(f) = f \otimes f \Delta_h(g) = g \otimes g.$$

Proof. Since that $\mathcal{U}_h(\mathcal{ST}(2))$ is a co-algebra with Δ_h, ε_h to prove that it is a Lie bialgebra it is enough to show that δ is the cobracket on $\mathcal{U}_h(\mathcal{ST}(2))$. That is, we must prove that δ satisfies cobracket properties (see Note 2). In fact, since $\delta(X_2) = 0$ we have that δ and Δ_h are compatible, that is, they satisfy trivially

$$\delta(X_iX_j) = \delta(X_1)\Delta_h(X_j) + \Delta_h(X_i)\delta(X_j) \ . \tag{2.15}$$

Since the co-Jacobi identity,

$$\mathcal{G} \xrightarrow{\delta} \mathcal{G}\otimes\mathcal{G} \xrightarrow{\delta\otimes\mathrm{id}} \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G} \xrightarrow{\Sigma} \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}$$

does not depend of Δ_h, when we extend δ over $\mathcal{U}_h(\mathcal{ST}(2))$, δ satisfies this property.

Likewise, δ satisfies the co-Leibniz identity ,

$$(\Delta_h\otimes id)\delta = (id\otimes\delta)\Delta_h + \sigma_{23}(\delta\otimes id)\Delta_h\ , \qquad (2.16)$$

where σ_{23} means the permutations of the last two elements. Since $\delta(X_2) = 0$ the identity is zero for X_2. While for X_1 the left side of (2.16) becomes

$$\begin{aligned}(\Delta_h\otimes\mathrm{id})\delta(X_1) &= (\Delta_h\otimes\mathrm{id})2(X_1\otimes X_2 - X_2\otimes X_1)\\ &= 2(\Delta_h(X_1)\otimes\mathrm{id}(X_2) - \Delta_h(X_2)\otimes\mathrm{id}(X_1))\\ &= 2(X_1\otimes f(h) + g(h)\otimes X_1)\otimes X_2 - (X_2\otimes f(h) + g(h)\otimes X_2)\otimes X_1)\\ &= 2(X_1\otimes f(h)\otimes X_2 + g(h)\otimes X_1\otimes X_2 - X_2\otimes f(h)\otimes X_1 - g(h)\otimes X_2\otimes X_1)\end{aligned}$$

and the right expression becomes

$$\begin{aligned}(\mathrm{id}\otimes\delta)\Delta_h(X_1)+\sigma_{23}(\delta\otimes\mathrm{id})\Delta_h(X_1) &= (\mathrm{id}\otimes\delta)(X_1\otimes f(h) + g(h)\otimes X_1) + \sigma_{23}(\delta\otimes\mathrm{id})(X_1\otimes f(h) + g(h)\otimes X_1)\\ &= X_1\otimes f(h)\delta(I) + g(h)\otimes\delta(X_1) + \sigma_{23}(\delta(X_1)\otimes f(h) + g(h)\delta(I)\otimes X_1)\\ &= 2g(h)\otimes(X_1\otimes X_2 - X_2\otimes X_1) + \sigma_{23}(X_1\otimes X_2 - X_2\otimes X_1)\otimes f(h))\\ &= 2(g(h)\otimes X_1\otimes X_2 - g(h)\otimes X_2\otimes X_1) + \sigma_{23}(X_1\otimes X_2\otimes f(h) - X_2\otimes X_1\otimes f(h))\\ &= 2(g(h)\otimes X_1\otimes X_2 - g(h)\otimes X_2\otimes X_1 + X_1\otimes f(h)\otimes X_2 - X_2\otimes f(h)\otimes X_1).\end{aligned}$$

Therefore $\mathcal{U}_h(\mathcal{ST}(2))$ has a Lie bialgebra structure with the bracket $[,]$ in (2.2), the cobracket δ and the coproduct $\{\Delta_h, \varepsilon_h\}$.

Since the quantum enveloping algebra is an algebra over formal power series in h, we can describe it by exponential expressions such as e^{hX_i}. To perform this description we need the following lemma:

Lemma 2.3. *The functions on* $\mathcal{U}_h(\mathcal{ST}(2))$

$$f(h) = e^{hX_2} \text{ y } g(h) = e^{-hX_2}$$

satisfy

$$\Delta_h(e^{hX_2}) = e^{hX_2}\otimes e^{hX_2}\ \ \Delta_h(e^{-hX_2}) = e^{-hX_2}\otimes e^{-hX_2}.$$

Proof. Since Δ_h is linear we can calculate Δ_h on the exponential function

$$e^{hX_2} = \sum_{n=0}^{\infty}\frac{h^n}{n!}X_2^n$$

to obtain

$$\begin{aligned}\Delta_h(e^{hX_2}) &= \sum_{n=0}^{\infty}\frac{h^n}{n!}(X_2\otimes I + I\otimes X_2)^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{h^n}{n!}X_2^k\otimes X_2^{n-k}\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{h^{k+m}}{k!m!}X_2^k\otimes X_2^m = e^{hX_2}\otimes e^{hX_2}.\end{aligned}$$

In a similar fashion we can prove the lemma for the function e^{-hX_2}.

Thus we can take in (2.14) $f(h) = e^{hX_2}$ and $g(h) = e^{-hX_2}$. Therefore, Δ_h is given by

$$\begin{aligned}\Delta_h(X_1) &= X_1\otimes e^{hX_2} + e^{-hX_2}\otimes X_1\\ \Delta_h(X_2) &= X_2\otimes e^{hX_2} + e^{-hX_2}\otimes X_2\\ \Delta_h([X_1,X_2]) &= [X_1,X_2]\otimes e^{hX_2} + e^{-hX_2}\otimes[X_1,X_2]\end{aligned}$$

Note 3. Since $\delta(X_2) = 0$ we must take the special expression for $\Delta_h(X_2)$. That is

$$\begin{aligned}\Delta_h(X_1) &= X_1\otimes e^{hX_2} + e^{-hX_2}\otimes X_1\\ \Delta_h(X_2) &= X_2\otimes I + I\otimes X_2 \qquad (2.17)\\ \Delta_h([X_1,X_2]) &= [X_1,X_2]\otimes e^{hX_2} + e^{-hX_2}\otimes[X_1,X_2]\end{aligned}$$

In this case δ in (2.4) is a cobracket on the $\mathcal{U}_h(\mathcal{ST}(2)$ that does not satisfy the co-Leibniz identity (2.16). However, $(\mathcal{U}_h(\mathcal{ST}(2), \Delta_h, \delta, \varepsilon, [,])$ is still a Lie bialgebra with Δ_h in (2.17).

With Δ_h defined in (2.17) we can find an antipode application on $\mathcal{U}_h(\mathcal{ST}(2)$. Let

$$S_h : \mathcal{U}_h(\mathcal{ST}(2)) \to \mathcal{U}_h(\mathcal{ST}(2))$$

be this application. Since S_h must satisfy the properties of the antipode application, it satisfies in particular the identity

$$m(S_h\otimes I)\Delta_h = m(I\otimes S_h)\Delta_h = 0$$

where m is the multiplication on $\mathcal{U}_h(\mathcal{ST}(2))$. Thus S_h satisfies the following lemma:

Lemma 2.4. *The application*

$$S_h : \mathcal{U}_h(\mathcal{ST}(2)) \to \mathcal{U}_h(\mathcal{ST}(2))$$

defined on the Lie bialgebra $(\mathcal{U}_h(\mathcal{ST}(2)), \Delta_h, \varepsilon_h, \delta)$ satisfies

$$m(S_h \otimes I)\Delta_h = m(I \otimes S_h)\Delta_h = 0 \ , \qquad (2.18)$$

if, and only if,

$$S_h(X_1) = -e^{hX_2} X_1 e^{-hX_2} \ , \quad S_h(X_2) = -X_2 \ . \quad (2.19)$$

Proof. The left-hand of (2.18) in X_1 takes the form

$$\begin{aligned} m(S_h \otimes I)\Delta_h(X_1) &= m(S_h \otimes I)(X_1 \otimes e^{hX_2} + e^{-hX_2} \otimes X_1) \\ &= m(S_h(X_1) \otimes e^{hX_2} + S_h(e^{-hX_2}) \otimes X_1) \\ &= S_h(X_1)e^{hX_2} + e^{hX_2}X_1 = 0 \end{aligned}$$

Thus

$$S_h(X_1) = -e^{hX_2} X_1 e^{-hX_2}$$

In the same way we can obtain $S_h(X_2) = -X_2$

Now we must prove that S_h defined in (2.19) is the antipode application on $\mathcal{U}_h(\mathcal{ST}(2))$.

Lemma 2.5. *The application S_h*

$$S_h : \mathcal{U}_h(\mathcal{ST}(2)) \to \mathcal{U}_h(\mathcal{ST}(2))$$

defined on $\{X_1, X_2\}$ by

$$S_h(X_1) = -e^{hX_2} X_1 e^{-hX_2} \ , \qquad S_h(X_2) = -X_2$$

is an antipode application on $\mathcal{U}_h(\mathcal{ST}(2))$.

Proof. In fact, S_h satisfies the following properties

$$\begin{aligned} m(S_h \otimes I)\Delta_h &= m(I \otimes S_h)\Delta_h = 0 \ , \\ S_h[X_1, X_2]_h &= -[S_h(X_1), S_h(X_2)]_h \ , \\ [X_i, S_h(X_i)]_h &= [S_h(X_i), X_i]_h \ , \quad \text{for} \quad i = 1, 2 \ . \end{aligned}$$

The first property is the property (2.18), which we used to find $S_h(X_1)$ and $S_h(X_2)$; thus S_h satisfies this property for X_1, X_2. The second property is obtained as a result of the following two expressions

$$\begin{aligned} S_h[X_1, X_2]_h &= S_h(2\,\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}}) \\ &= 2\,\frac{e^{-hX_2} - e^{hX_2}}{(e^h - e^{-h})} \\ &= -2\,\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}} \ , \end{aligned}$$

$$\begin{aligned} [S_h(X_1), S_h(X_2)]_h &= [-e^{hX_2} X_1 e^{-hX_2}, -X_2]_h \\ &= e^{hX_2}[X_1, X_2]_h e^{-hX_2} \\ &= e^{hX_2}(2\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}})e^{-hX_2} \\ &= 2\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}} \ , \end{aligned}$$

where we have used that $S(e^{hX_2}) = e^{-hX_2}$ and $S(e^{-hX_2}) = e^{hX_2}$ from the definition of the exponential function. In the same form we prove the last property of antipode,

$$\begin{aligned} [X_1, S_h(X_1)]_h &= [X_1, -e^{hX_2} X_1 e^{-hX_2}]_h \\ &= -e^{hX_2}\ [X_1, X_1]_h\ e^{-hX_2} = 0 \ . \\ [S_h(X_1), X_1]_h &= [-e^{hX_2} X_1 e^{-hX_2}, X_1]_h \\ &= -e^{hX_2}\ [X_1, X_1]_h\ e^{-hX_2} = 0 \ . \end{aligned}$$

The Lemmas (2.4) and (2.5) complete the proof of the following proposition:

Proposition 2.3. *The algebra $\mathcal{U}_h(\mathcal{ST}(2))$ generated by X_1, X_2, I with the operations defined by*

$$\begin{aligned} [X_1, X_2]_h &= 2\,\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}} \\ \Delta_h(X_1) &= X_1 \otimes e^{hX_2} + e^{-hX_2} \otimes X_1 \\ \Delta_h(X_2) &= X_2 \otimes I + I \otimes X_2 \\ \Delta_h([X_1, X_2]) &= [X_1, X_2] \otimes e^{hX_2} + \otimes[X_1, X_2] \\ \varepsilon_h(X_1) &= \varepsilon_h(X_2) = 0 \\ S_h(X_1) &= -e^{hX_2} X_1 e^{-hX_2} \\ S_h(X_2) &= S(X_2) = -X_2. \end{aligned}$$

has the structure of a Hopf algebra.

Since when $h \to 0$, the coalgebra structure of $\mathcal{U}_h(\mathcal{ST}(2))$ coincides with the bialgebra $\mathcal{ST}(2)$.

Finally, we verify the $*$−algebra structure.

Proposition 2.4. *The algebra $\mathcal{U}_h(\mathcal{ST}(2))$ is a Hopf $*$−algebra with $X_1 = X_1^*, X_2 = X_2^*$.*

Proof. Let $* : X_i \to X_i^*$, $i = 1, 2$ be the involution. Then the operations defined in the proposition (2.3) are $*$−algebra maps. In fact, we have

$$(e^{hX_2})^* = e^{hX_2}, \quad (e^{-hX_2})^* = e^{-hX_2}$$

then

$$[X_1^*, X_2^*]_h = [X_1, X_2]_h = 2\,\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}}$$
$$= 2\left(\frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}}\right)^* = [X_1, X_2]_h^*$$

Similary we can see that Δ and ε are $*$-algebra maps.

Because of propositions (2.4) and (2.3) we can affirm that the Hopf algebra $\mathcal{U}_h(\mathcal{ST}(2))$ is the quantum group of the universal enveloping algebra $\mathcal{U}(\mathcal{ST}(2))$.

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