ON THE QUANTUM STRUCTURE OF THE
UNIVERSAL ENVELOPING ALGEBRA OF
THE LIE ALGEBRA $ST(2)$

by

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Abstract


The structure of Hopf co-Poisson algebra on the universal enveloping algebra $U(ST(2))$ of Lie algebra $ST(2)$ is determined with the help of a solution of the Yang–Baxter equation. Using this solution, a bracket on the dual space of Lie algebra $ST(2)$ is also determined. This cobracket on $ST(2)$ induces a deformation of the universal enveloping algebra $U(ST(2))$ which has a Hopf algebra structure, as we shall verify. This Hopf algebra is called the quantum group associated to a universal enveloping algebra.

Key words: Lie bialgebras, Hopf algebras, Poisson brackets, Lie Poisson group, Hopf co-Poisson algebra, Universal enveloping algebra, $r$–matrix, Quantum group, Yang–Baxter equation.

Resumen

Con ayuda de una solución de la ecuación clásica de Yang–Baxter determinamos la estructura de álgebra de Hopf-co-Poisson del álgebra envolvente universal $U(ST(2))$ del álgebra de Lie $ST(2)$. Usando esta solución determinamos un cochete en el espacio dual del álgebra de Lie. Este co-corchete sobre $ST(2)$ induce una deformación del álgebra envolvente universal $U((ST(2))$ que tiene estructura de álgebra de Hopf, como probaremos. Esta álgebra de Hopf es llamada el grupo cuántico asociado a un álgebra envolvente universal.


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1. Introduction

Let $G$ be a Lie Group and $\mathcal{G}$ its Lie algebra; we can obtain quantum groups as deformations of the algebra of $C^\infty$ functions $\mathcal{F}(G)$ on $G$, or as quantizations of a Lie bialgebra $\mathcal{G}$. A quantization of a Lie bialgebra $\mathcal{G}$ is a deformation of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ equipped with the co-Poisson Hopf algebra structure, such that the classical limit of this quantization is the Lie bialgebra structure of $\mathcal{G}$. To construct a deformation of the universal enveloping algebra we need to describe the co-Poisson Hopf algebra structure on $\mathcal{U}(\mathcal{G})$ or, equivalently, we must build the bialgebra structure on the Lie algebra $\mathcal{G}$.

The purpose of this work is to describe a mathematical procedure to produce a quantum group structure associated to a universal enveloping algebra, the universal enveloping algebra $\mathcal{U}(ST(2))$ of Lie algebra $ST(2)$. To achieve this purpose we use a solution of the classical Yang–Baxter equation (CYBE) on Lie algebra $ST(2)$. We build a cobracket in $ST(2)$ connected with this solution. This cobracket determines a bialgebra structure in $ST(2)$ and the co-Poisson-Hopf algebra structure in the universal enveloping algebra $\mathcal{U}(ST(2))$ as we shall verify.

We will deform the comultiplication in $\mathcal{U}(ST(2))$ by means of a parameter $h$ in order to build the algebra $\mathcal{U}_h(ST(2))$. We shall find appropriate expressions for the coproduct $\Delta_h$, the antipode application $S_h$ and the bracket $[ ]_h$ on $\mathcal{U}_h(ST(2))$. We shall prove that $\mathcal{U}_h(ST(2))$, with these applications, has a Hopf algebra structure so that when $h \rightarrow 0$ the coalgebra structure of $\mathcal{U}_h(ST(2))$ coincides with the bialgebra structure of $ST(2)$. That is, we shall prove that this algebra is a quantum group of the universal enveloping algebra type.

2. The group $ST(2)$ and its algebra $\mathcal{U}(ST(2))$

2.1. The Lie algebra $ST(2)$

Let $ST(2)$ be the Lie group of upper triangular matrices $2 \times 2$ with determinant equal to 1 such that the operation of the group is the multiplication between matrices.

The Lie algebra $ST(2)$ associated to $ST(2)$ is the Lie algebra of upper triangular matrices $2 \times 2$ with null trace on $\mathbb{R}$, where the matrices

$$
X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
$$

form a basis with the Lie bracket given by

\begin{align*}
[X_1, X_2] &= -[X_2, X_1] = 2X_2, \\
[X_i, X_i] &= 0, \quad i = 1, 2
\end{align*}

2.2. The Lie bialgebra structure on $ST(2)$

A Lie bialgebra is a Lie algebra with a Lie co-algebra structure $\delta$ fulfilling the 1-cocycle condition (2.5) with respect to the tensorial adjoint representation, (see [6] p. 43).

The Lie bialgebra structures may be induced by the adjoint application of Lie algebras

$$
ad : ST(2) \rightarrow ST(2) \times ST(2)
$$

defined by

$$
ad_X(Y) = [X, Y], \quad \text{for } X, Y \in ST(2).
$$

The adjoint representation of $ST(2)$ is totally determined by its representation on the $\{X_1, X_2\}$ basis of Lie algebra, given by

\begin{align*}
ad_{X_1}(X_1) &= [X_1, X_1] = 0 \\
ad_{X_1}(X_2) &= [X_1, X_2] = 2X_2 \\
ad_{X_2}(X_1) &= [X_2, X_1] = -2X_2 \\
ad_{X_2}(X_2) &= 0.
\end{align*}

Any representation of a Lie algebra can be extended to a unique representation on the tensorial product of Lie algebras. Then we can extend the adjoint representation just defined to the adjoint tensorial representation in the following way

$$
(ad_X \otimes I + I \otimes ad_X) : ST(2) \otimes ST(2) \rightarrow ST(2) \otimes ST(2)
$$

\begin{align*}
(ad_X \otimes I + I \otimes ad_X)(X_i \otimes X_j) \\
&= (ad_X \otimes I + I \otimes ad_X)(X_i \otimes X_j) \\
&= (ad_X(X_i) \otimes IX_j + IX_i \otimes ad_X(X_j)) \\
&= [X, X_i] \otimes X_j + X_i \otimes [X, X_j].
\end{align*}
With this adjoint tensorial representation and with one special $\tau$-tensor, $r \in ST(2) \otimes ST(2)$, we are able to define a co-Lie algebra structure.

An element $r \in ST(2) \otimes ST(2)$ defines a Lie bialgebra structure if and only if $r$ is skew-symmetric and $[[r, r]] = 0$. The equation $[[r, r]] = 0$ is called the classical Yang-Baxter equation (CYBE). Moreover if $r$ is a skew-symmetric element, $r$ is called an $\tau$-matrix.

We know that the Lie algebra $ST(2)$ has only one $\tau$-matrix (see [9]), and that this $\tau$-matrix is the tensor

$$ r = X_1 \otimes X_2 - X_2 \otimes X_1, \quad (2.3) $$

$X_1, X_2$ being the basis elements of $ST(2)$. Thus $r$ is skew-symmetric and satisfies the Yang Baxter equation,

$$ [[r, r]] = [r_{12} + r_{13}] + [r_{12} + r_{23}] + [r_{13} + r_{23}] = 0. $$

where

$$ r_{12} = X_1 \otimes X_2 \otimes I - X_2 \otimes X_1 \otimes I $$

$$ r_{23} = I \otimes X_1 \otimes X_2 - I \otimes X_2 \otimes X_1 $$

$$ r_{13} = X_1 \otimes I \otimes X_2 - X_2 \otimes I \otimes X_1 $$

**Proposition 2.1.** The $\tau$-matrix

$$ r = X_1 \otimes X_2 - X_2 \otimes X_1 $$

induces a cobracket on the Lie algebra $ST(2)$ by the application

$$ \delta : ST(2) \rightarrow ST(2) \otimes ST(2) $$

defined by

$$ \delta(X) = (ad_X \otimes I + I \otimes ad_X)(r) = X \cdot r, $$

for $X \in ST(2)$.

**Proof.** We can show that $\delta$ satisfies the properties of a cobracket. For this purpose it is enough to prove that $\delta$ satisfies these properties on $\{X_1, X_2\}$, the basis of the Lie algebra. In those elements, $\delta$ is given by

$$ \delta(X_1) = X_1 \cdot r = (ad_{X_1} \otimes I + I \otimes ad_{X_1})(X_1 \otimes X_2 - X_2 \otimes X_1) $$

$$ = -[X_1, X_2] \otimes X_1 + X_1 \otimes [X_1, X_2] $$

$$ = 2(X_1 \otimes X_2 - X_2 \otimes X_1), $$

$$ \delta(X_2) = X_2 \cdot r = (ad_{X_2} \otimes I + I \otimes ad_{X_2})(X_1 \otimes X_2 - X_2 \otimes X_1) $$

$$ = [X_2, X_1] \otimes X_2 - X_2 \otimes [X_2, X_1] = 0. \quad (2.4) $$

Then $\delta$ satisfies the following cobracket properties over $\{X_1, X_2\}$:

1. If $\delta(X) = \sum X_i \otimes X_j$ then

$$ X_i \otimes X_j = -X_j \otimes X_i \text{ for all } i, j. $$

2. $\delta$ satisfies the associative property,

$$ (\text{id} \otimes \delta) \circ \delta - (\delta \otimes \text{id}) \circ \delta = 0. $$

3. $\delta$ is one 1-cocycle, that is, $\delta$ satisfies

$$ \delta([X_i, X_j]) $$

$$ = (ad_{X_i} \otimes I + I \otimes ad_{X_i})(\delta(X_j)) - (ad_{X_j} \otimes I + I \otimes ad_{X_j})(\delta(X_i)) $$

$$ = X_i \cdot \delta(X_j) - X_j \cdot \delta(X_i) $$

(2.5)

with $i, j = 1, 2$ and $X_i, X_j \in ST(2)$.

The proof of these properties is straightforward. However, since $\delta$ is defined by an $\tau$-matrix we can deduce, from Proposition 2.1.2 of [2], that $\delta$ is a cobracket on $(ST(2))$. Therefore $(ST(2), \delta)$ is a Lie bialgebra.

The Lie bialgebra $(ST(2), \delta)$ is called a quasitriangular bialgebra because it is generated by a solution of the CYBE and triangular bialgebra because it arises from a skew-symmetric solution of the CYBE. This Lie bialgebra is also called a coboundary Lie bialgebra because the cobracket $\delta$ is a 1-cocycle.

### 2.3. The universal enveloping algebra $U(ST(2))$

Let $ST(2)$ be the Lie algebra of $ST(2)$ and let $T(ST(2))$ be the tensorial algebra of $ST(2)$,

$$ T(ST(2)) = \oplus_{n \geq 0} T^n(ST(2)) = \oplus_{n \geq 0} (ST(2))^{\otimes n} $$

where

$$ T^0 ST(2) = (ST(2))^0 = k, \quad T^1 ST(2) = (ST(2)), $$

$$ T^n(ST(2)) = (ST(2)) \otimes (ST(2)) \otimes \cdots \otimes (ST(2)). $$

The universal enveloping algebra $U(ST(2))$ of the Lie algebra $ST(2)$ is the associative algebra,

$$ U(ST(2)) \cong T(ST(2))/I, $$

where $I$ is the ideal of $T(ST(2))$ engendered by

$$ X_1 \otimes X_2 - X_2 \otimes X_1 - 2X_2, \quad X_1, X_2 \in ST(2), $$

with the product given by the recurrent operation,

$$ (X_1 \otimes X_2^n)(X_1^p X_2^q) = X_1^{m-1} X_1 X_2 X_2^{n-1} X_1^p X_2^q $$

$$ = X_1^{m-1} (2X_2 + X_2 X_1) X_2^{n-1} X_1^p X_2^q $$

$$ = 2X_1^{m-1} X_2 X_1^p X_2^q + X_1^{m-1} X_2 X_1^p X_2^q $$

$$ = \cdots $$

**Note 1.** It is known that a universal enveloping algebra $U(G)$ of the Lie algebra $G$ is a Hopf algebra (see [2],...
2.4. The Quantum algebra $\mathcal{U}_h(S\mathcal{T}(2))$

The quantum enveloping algebra $\mathcal{U}_h(S\mathcal{T}(2))$ of the Lie bialgebra $(S\mathcal{T}(2))$ is a quantization of $\mathcal{U}(S\mathcal{T}(2))$ when $\mathcal{U}(S\mathcal{T}(2))$ is considered as a co-Poisson-Hopf algebra. It means that $(S\mathcal{T}(2))$ has a quasitriangular structure (see [4]). Since the bialgebra $(S\mathcal{T}(2))$ has a quasitriangular structure we need work with exponential functions, it means that we should work over the ring $\mathbb{R}[[h]]$ of the formal series in $\hbar$.

In order to build a quantization of $\mathcal{U}(S\mathcal{T}(2))$ we consider the Lie algebra $\mathcal{U}(S\mathcal{T}(2))[[h]] = \mathcal{U}_h(S\mathcal{T}(2))$ of formal power series in $\mathbb{R}[[h]]$ with coefficients in $\mathcal{U}(S\mathcal{T}(2))$, generated by $X_1, X_2, I$ with the defining relation

$$[X_1, X_2]_h = 2 \frac{e^h X_2 - e^{-h} X_1}{e^h - e^{-h}}$$

and

$$[X_i, X_i]_h = 0, \quad \text{for } i = 1, 2.$$  

We must prove that $\mathcal{U}(S\mathcal{T}(2))[[h]]$ with this new product, has a Hopf $*$-algebra structure, so that when $h \to 0$ the coalgebra structure of $\mathcal{U}_h(S\mathcal{T}(2))$ coincides with the bialgebra structure of $S\mathcal{T}(2)$.

We shall find appropriate expressions for the coproduct $\Delta_h$, the counit $\epsilon_h$ and the antipode application $S_h$ defined on $\mathcal{U}(S\mathcal{T}(2))[[h]]$. For this purpose we take $\Delta_h$ as the deformation of co-product $\Delta$ defined on $\mathcal{U}(S\mathcal{T}(2))$. Let be $\Delta_h$ the linear application

$$\Delta_h : \mathcal{U}_h(S\mathcal{T}(2)) \to \mathcal{U}_h(S\mathcal{T}(2)) \otimes \mathcal{U}_h(S\mathcal{T}(2))$$

defined by

$$\Delta_h = \Delta + \frac{h}{2} \delta + O(h^2)$$

such that $\lim_{h \to 0} \Delta_h \to \Delta$. Here $\delta$ is the cobracket (2.4) and $\hbar$ is the parameter of the deformation. The map $\Delta_h$ in $X_1, X_2$ is given by

$$\Delta_h(X_1) = X_1 \otimes (I + hX_2) + (I - hX_2) \otimes X_1$$
$$\Delta_h(X_2) = X_2 \otimes I + I \otimes X_2.$$  

(2.8)

Since $(I + hX_2)$ and $(I - hX_2)$ are functions in $\mathcal{U}(S\mathcal{T}(2))[[h]]$ we can write more generally,

$$\Delta_h(X_1) = X_1 \otimes f(h) + g(h) \otimes X_1$$
$$\Delta_h(X_2) = X_2 \otimes f(h) + g(h) \otimes X_2.$$  

(2.9)

and since we must have $\lim_{h \to 0} f(h) = I, \lim_{h \to 0} g(h) = I$.
For these two functions we have:

**Lemma 2.1.** For any choice of \( f \) and \( g \), the application

\[
\Delta_h : \mathcal{U}_h(\mathcal{S}(2)) \to \mathcal{U}_h(\mathcal{S}(2)) \otimes \mathcal{U}_h(\mathcal{S}(2))
\]

is defined by (2.9) is an homomorphism of Lie algebras.

**Proof.** In fact \( \Delta_h \) satisfies

\[
\Delta_h([X_1, X_2]) = [\Delta_h(X_1), \Delta_h(X_2)]
\]

whereas

\[
[\Delta_h(X_1), \Delta_h(X_2)] = \Delta_h(X_1)\Delta_h(X_2) - \Delta_h(X_2)\Delta_h(X_1)
\]

\[
= (X_1 \otimes f(h) + g(h) \otimes X_1)(X_2 \otimes f(h) + g(h) \otimes X_2) - (X_2 \otimes f(h) + g(h) \otimes X_2)(X_1 \otimes f(h) + g(h) \otimes X_1)
\]

\[
= (X_1X_2 \otimes f(h) + g(h) \otimes X_1X_2 + X_1 \otimes f(h)X_2 + g(h)X_2 \otimes X_1)
\]

\[
- (X_2X_1 \otimes f(h) + g(h) \otimes X_2X_1 + X_2g(h) \otimes X_1 + X_1 \otimes X_2f(h))
\]

\[
= [X_1, X_2] \otimes f(h) + g(h) \otimes [X_1, X_2] = \Delta_h([X_1, X_2]).
\]

**Lemma 2.2.** The application \( \Delta_h \) defined by (2.9) is coassociative if and only if \( f \) and \( g \) satisfy

\[
\Delta_h(f) = f \otimes f, \quad \Delta_h(g) = g \otimes g
\]

**Proof.** We must proof that \( \Delta_h \) satisfies

\[
(\Delta_h \otimes \text{id})\Delta_h(X) = (\text{id} \otimes \Delta_h)\Delta_h(X)
\]

if and only if, \( f \) and \( g \) satisfy

\[
\Delta_h(f) = f \otimes f, \quad \Delta_h(g) = g \otimes g
\]

In fact, the right-hand side of (2.11) takes the form

\[
(\Delta_h \otimes \text{id})\Delta_h(X_i) = (\Delta_h \otimes \text{id})(X_i \otimes f(h) + g(h) \otimes X_i)
\]

\[
= \Delta_h(X_i) \otimes f(h) + \Delta_h(g(h)) \otimes X_i
\]

\[
= X_i \otimes f(h) + g(h) \otimes X_i \otimes f(h)
\]

\[
+ \Delta_h(g(h)) \otimes X_i \quad (i = 1, 2)
\]

and the left-hand side of (2.11) takes the form

\[
(\text{id} \otimes \Delta_h)\Delta_h(X_i) = (\text{id} \otimes \Delta_h)(X_i \otimes f(h) + g(h) \otimes X_i)
\]

\[
= X_i \otimes \Delta_h(f(h)) + g(h) \otimes \Delta_h(X_i)
\]

\[
= X_i \otimes \Delta_h(f(h)) + g(h) \otimes (X_i \otimes f(h) + g(h) \otimes X_i)
\]

\[
= X_i \otimes \Delta_h(f(h)) + g(h) \otimes X_i \otimes f(h)
\]

\[
+ g(h) \otimes g(h) \otimes X_i
\]

We see that this two expressions are equal if, and only if,

\[
\Delta_h(f(h)) = f(h) \otimes f(h) \quad \text{and} \quad \Delta_h(g(h)) = g(h) \otimes g(h)
\]

If we take the application

\[
\varepsilon_h : \mathcal{U}_h(\mathcal{S}(2)) \to \mathbb{R},
\]

defined by

\[
\varepsilon_h(X_1) = \varepsilon_h(X_2) = 0,
\]

then \( \varepsilon_h \) is a counit for the application \( \Delta_h \) in (2.9), that is, \( \Delta_h \) and \( \varepsilon_h \) satisfy

\[
\Delta_h(\text{id} \otimes \varepsilon) = \Delta_h(\varepsilon \otimes \text{id})
\]

trivially. We can infer from Lemmas (2.1) and (2.2) that the map \( \Delta_h \) in (2.9), with \( \varepsilon_h \) in (2.13), is a co-product on the space \( \mathcal{U}_h(\mathcal{S}(2)) \). Besides, if we consider the map \( \delta \) in (2.4) extended on \( \mathcal{U}_h(\mathcal{S}(2)) \) we have the following assertion:

**Proposition 2.2.** The space \( \mathcal{U}_h(\mathcal{S}(2)) \) generate by \( X_1, X_2 \) is a Lie bialgebra, with the coproduct \( \Delta_h \) the counit \( \varepsilon_h \) and the cobracket \( \delta \) define by

\[
\Delta_h(X_1) = X_1 \otimes f(h) + g(h) \otimes X_1
\]

\[
\Delta_h(X_2) = X_2 \otimes f(h) + g(h) \otimes X_2
\]

\[
\varepsilon_h(X_1) = \varepsilon_h(X_2) = 0
\]

\[
\delta(X_1) = 2(X_1 \otimes X_2 - X_2 \otimes X_1)
\]

\[
\delta(X_2) = 0
\]

where \( f \) and \( g \) are functions that satisfy the following properties:

\[
\lim_{h \to 0} f(h) = I, \quad \lim_{h \to 0} g(h) = I
\]

\[
\Delta_h(f) = f \otimes f \Delta_h(g) = g \otimes g.
\]

**Proof.** Since that \( \mathcal{U}_h(\mathcal{S}(2)) \) is a co-algebra with \( \Delta_h, \varepsilon_h \) to prove that it is a Lie bialgebra it is enough to show that \( \delta \) is the cobracket on \( \mathcal{U}_h(\mathcal{S}(2)) \). That is, we must prove that \( \delta \) satisfies cobracket properties (see Note 2). In fact, since \( \delta(X_2) = 0 \) we have that \( \delta \) and \( \Delta_h \) are compatible, that is, they satisfy trivially

\[
\delta(X_i X_j) = \delta(X_i) \Delta_h(X_j) + \Delta_h(X_i) \delta(X_j).
\]
Since the co-Jacobi identity,
\[ \delta \circ (\delta \circ \id) = (\delta \circ \id) \circ \delta \]
does not depend of \( \Delta_h \), when we extend \( \delta \) over \( \mathcal{U}_h(ST(2)) \), \( \delta \) satisfies this property.

\[ (\Delta_h \circ \id) \delta(X_1) = (\Delta_h \circ \id) 2(X_1 \otimes X_2 - X_2 \otimes X_1) \]
\[ = 2(\Delta_h(X_1) \otimes \id(X_2) - \Delta_h(X_2) \otimes \id(X_1)) \]
\[ = 2(X_1 \otimes f(h) + g(h) \otimes X_1) \otimes X_2 - (X_2 \otimes f(h) + g(h) \otimes X_2) \otimes X_1) \]
\[ = 2(X_1 \otimes f(h) \otimes X_2 + g(h) \otimes X_1 \otimes X_2 - X_2 \otimes f(h) \otimes X_1 - g(h) \otimes X_2 \otimes X_1) \]

and the right expression becomes

\[ (\id \circ \delta) \Delta_h(X_1) + \sigma_{23} (\delta \circ \id) \Delta_h(X_1) = (\id \circ \delta)(X_1 \otimes f(h) + g(h) \otimes X_1) + \sigma_{23}(\delta \circ \id)(X_1 \otimes f(h) + g(h) \otimes X_1) \]
\[ = X_1 \otimes f(h)\delta(I) + g(h) \otimes X_1 \delta(I) + \sigma_{23}(X_1 \otimes (X_1 \otimes X_2 - X_2 \otimes X_1)) \]
\[ = 2(g(h) \otimes (X_1 \otimes X_2 - X_2 \otimes X_1)) + \sigma_{23}(X_1 \otimes X_2 - X_2 \otimes X_1) \otimes f(h) \]
\[ = 2(g(h) \otimes X_1 \otimes X_2 - g(h) \otimes X_2 \otimes X_1) + \sigma_{23}(X_1 \otimes X_2 \otimes f(h) - X_2 \otimes X_1 \otimes f(h)) \]
\[ = 2(g(h) \otimes X_1 \otimes X_2 - g(h) \otimes X_2 \otimes X_1 + X_1 \otimes f(h) \otimes X_2 - X_2 \otimes f(h) \otimes X_1). \]

Therefore \( \mathcal{U}_h(ST(2)) \) has a Lie bialgebra structure with the bracket \([,] \) in (2.2), the cobracket \( \delta \) and the coproduct \( \{ \Delta_h, \varepsilon_h \} \).

Since the quantum enveloping algebra is an algebra over formal power series in \( h \), we can describe it by exponential expressions such as \( e^{hX_1} \). To perform this description we need the following lemma:

**Lemma 2.3.** The functions on \( \mathcal{U}_h(ST(2)) \)

\[ f(h) = e^{hX_2} \text{ and } g(h) = e^{-hX_2} \]

satisfy

\[ \Delta_h(e^{hX_2}) = e^{hX_2} \otimes e^{hX_2} = \Delta_h(e^{-hX_2}) = e^{-hX_2} \otimes e^{-hX_2}. \]

**Proof.** Since \( \Delta_h \) is linear we can calculate \( \Delta_h \) on the exponential function

\[ e^{hX_2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} X_2^n \]

to obtain

\[ \Delta_h(e^{hX_2}) = \sum_{n=0}^{\infty} \frac{h^n}{n!} (X_2 \otimes I + I \otimes X_2)^n \]
\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{h^n}{n!} X_2^k \otimes X_2^{n-k} \]
\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{k+n} \frac{h^{k+m}}{k!m!} X_2^k \otimes X_2^m = e^{hX_2} \otimes e^{hX_2}. \]

In a similar fashion we can prove the lemma for the function \( e^{-hX_2} \).

Likewise, \( \delta \) satisfies the co-Leibniz identity,

\[ (\Delta_h \circ \id) \delta = (\id \circ \delta) \Delta_h + \sigma_{23}(\delta \circ \id) \Delta_h, \quad (2.16) \]

where \( \sigma_{23} \) means the permutations of the last two elements. Since \( \delta(X_2) = 0 \) the identity is zero for \( X_2 \). While for \( X_1 \) the left side of (2.16) becomes

\[ 2(g(h) \otimes X_1 \otimes X_2 - g(h) \otimes X_2 \otimes X_1) + \sigma_{23}(X_1 \otimes X_2 \otimes f(h) - X_2 \otimes X_1 \otimes f(h)) \]
\[ = 2(g(h) \otimes X_1 \otimes X_2 - g(h) \otimes X_2 \otimes X_1 + X_1 \otimes f(h) \otimes X_2 - X_2 \otimes f(h) \otimes X_1). \]

Thus we can take in (2.14) \( f(h) = e^{hX_2} \) and \( g(h) = e^{-hX_2} \). Therefore, \( \Delta_h \) is given by

\[ \Delta_h(X_1) = X_1 \otimes e^{hX_2} + e^{-hX_2} \otimes X_1 \]
\[ \Delta_h(X_2) = X_2 \otimes e^{hX_2} + e^{-hX_2} \otimes X_2 \]
\[ \Delta_h([X_1, X_2]) = [X_1, X_2] \otimes e^{hX_2} + e^{-hX_2} \otimes [X_1, X_2] \]

**Note 3.** Since \( \delta(X_2) = 0 \) we must take the special expression for \( \Delta_h(X_2) \). That is

\[ \Delta_h(X_1) = X_1 \otimes e^{hX_2} + e^{-hX_2} \otimes X_1 \]
\[ \Delta_h(X_2) = X_2 \otimes I + I \otimes X_2 \]
\[ \Delta_h([X_1, X_2]) = [X_1, X_2] \otimes e^{hX_2} + e^{-hX_2} \otimes [X_1, X_2] \]

In this case \( \delta \) in (2.4) is a cobracket on the \( \mathcal{U}_h(ST(2)) \) that does not satisfy the co-Leibniz identity (2.16). However, \( \{ \mathcal{U}_h(ST(2), \Delta_h, \delta, \varepsilon, [,] \} \) is still a Lie bialgebra with \( \Delta_h \) in (2.17).

With \( \Delta_h \) defined in (2.17) we can find an antipode application on \( \mathcal{U}_h(ST(2)) \). Let

\[ S_h : \mathcal{U}_h(ST(2)) \rightarrow \mathcal{U}_h(ST(2)) \]

be this application. Since \( S_h \) must satisfy the properties of the antipode application, it satisfies in particular the identity

\[ m(S_h \otimes I) \Delta_h = m(I \otimes S_h) \Delta_h = 0 \]

where \( m \) is the multiplication on \( \mathcal{U}_h(ST(2)) \). Thus \( S_h \) satisfies the following lemma:
Lemma 2.4. The application

\[ S_h : \mathcal{U}_h(ST(2)) \rightarrow \mathcal{U}_h(ST(2)) \]

defined on the Lie bialgebra \((\mathcal{U}_h(ST(2)), \Delta_h, \varepsilon_h, \delta)\) satisfies

\[ m(S_h \otimes I) \Delta_h = m(I \otimes S_h) \Delta_h = 0 , \quad (2.18) \]

if, and only if,

\[ S_h(X_1) = -e^{hX_2}X_1e^{-hX_2} , \quad S_h(X_2) = -X_2 . \quad (2.19) \]

Proof. The left-hand of (2.18) in \(X_1\) takes the form

\[
m(S_h \otimes I) \Delta_h(X_1) = m(S_h \otimes I)(X_1 \otimes e^{hX_2} + e^{-hX_2} \otimes X_1)
= m(S_h(X_1) \otimes e^{hX_2} + S_h(e^{-hX_2}) \otimes X_1)
= S_h(X_1)e^{hX_2} + e^{hX_2}X_1 = 0
\]

Thus

\[ S_h(X_1) = -e^{hX_2}X_1e^{-hX_2} \]

In the same way we can obtain \(S_h(X_2) = -X_2\).

Now we must prove that \(S_h\) defined in (2.19) is the antipode application on \(\mathcal{U}_h(ST(2))\).

Lemma 2.5. The application \(S_h\)

\[ S_h : \mathcal{U}_h(ST(2)) \rightarrow \mathcal{U}_h(ST(2)) \]

defined on \(\{X_1, X_2\}\) by

\[ S_h(X_1) = -e^{hX_2}X_1e^{-hX_2} , \quad S_h(X_2) = -X_2 \]

is an antipode application on \(\mathcal{U}_h(ST(2))\).

Proof. In fact, \(S_h\) satisfies the following properties

\[ m(S_h \otimes I) \Delta_h = m(I \otimes S_h) \Delta_h = 0 , \]

\[ S_h[X_1, X_2]_h = -[S_h(X_1), S_h(X_2)]_h , \]

\[ [X_i, S_h(X_i)]_h = [S_h(X_i), X_i]_h , \quad \text{for} \quad i = 1, 2 . \]

The first property is the property (2.18), which we used to find \(S_h(X_1)\) and \(S_h(X_2)\); thus \(S_h\) satisfies this property for \(X_1, X_2\). The second property is obtained as a result of the following two expressions

\[
S_h[X_1, X_2]_h = S_h(2 \frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}})
= 2 \frac{e^{-hX_2} - e^{hX_2}}{(e^h - e^{-h})}
= -2 \frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}},
\]

where we have used that \(S(e^{hX_2}) = e^{-hX_2}\) and \(S(e^{-hX_2}) = e^{hX_2}\) from the definition of the exponential function. In the same form we prove the last property of antipode,

\[
[X_1, S_h(X_1)]_h = [X_1, -e^{hX_2}X_1e^{-hX_2}]_h
= -e^{hX_2} [X_1, X_1]_h e^{-hX_2} = 0 .
\]

\[
[S_h(X_1), X_1]_h = [-e^{hX_2}X_1e^{-hX_2}, X_1]_h
= -e^{hX_2} [X_1, X_1]_h e^{-hX_2} = 0 .
\]

The Lemmas (2.4) and (2.5) complete the proof of the following proposition:

Proposition 2.3. The algebra \(\mathcal{U}_h(ST(2))\) generated by \(X_1, X_2, I\) with the operations defined by

\[ [X_1, X_2]_h = 2 \frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}} \]

\[ \Delta_h(X_1) = X_1 \otimes e^{hX_2} + e^{-hX_2} \otimes X_1 \]

\[ \Delta_h(X_2) = X_2 \otimes I + I \otimes X_2 \]

\[ \Delta_h([X_1, X_2]) = [X_1, X_2] \otimes e^{hX_2} + \otimes [X_1, X_2] \]

\[ \varepsilon_h(X_1) = \varepsilon_h(X_2) = 0 \]

\[ S_h(X_1) = -e^{hX_2}X_1e^{-hX_2} \]

\[ S_h(X_2) = S(X_2) = -X_2 . \]

has the structure of a Hopf algebra.

Since when \(h \rightarrow 0\), the coalgebra structure of \(\mathcal{U}_h(ST(2))\) coincides with the bialgebra \(ST(2)\).

Finally, we verify the \(*-\)algebra structure.

Proposition 2.4. The algebra \(\mathcal{U}_h(ST(2))\) is a Hopf \(*-\)algebra with \(X_1 = X_1^* , X_2 = X_2^* \).

Proof. Let \(* : X_i \rightarrow X_i^*, \ i = 1, 2\) be the involution. Then the operations defined in the proposition (2.3) are \(*-\)algebra maps. In fact, we have

\[
(e^{hX_2})^* = e^{hX_2}, \quad (e^{-hX_2})^* = e^{-hX_2}
\]
then

\[ [X_1^*, X_2^*]_h = [X_1, X_2]_h = 2 \frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}} \]

\[ = 2 \left( \frac{e^{hX_2} - e^{-hX_2}}{e^h - e^{-h}} \right)^* = [X_1, X_2]^*_h \]

Similarly we can see that \( \Delta \) and \( \varepsilon \) are *-algebra maps.

Because of propositions (2.4) and (2.3) we can affirm that the Hopf algebra \( U_h(ST(2)) \) is the quantum group of the universal enveloping algebra \( U(ST(2)) \).

References

[9] Berenice Guerrero, Quantización no estándar del grupo triangular \( ST(S) \), Lecturas Matemáticas 18 (1997), 23–44.