A NOTE ON THE CAUCHY PROBLEM OF FUZZY DIFFERENTIAL EQUATIONS

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Abstract

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In this paper we analyze the existence and uniqueness of solutions for a fuzzy initial value problem of kind $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, where $f : T \times X \to X$ is a fuzzy-valued mapping, T is a time interval, X is a class of fuzzy sets, $x_0 \in X$ and $t_0 \in T$. We consider $x'(t)$ as a generalization of the Hukuhara derivative.

Key words: Fuzzy-valued Mappings, Fuzzy Differentiability, Generalized Hukuhara Derivative, Fuzzy Differential Equations, Fuzzy Cauchy Problem.

Resumen

En este artículo se analiza la existencia y unicidad de soluciones para el siguiente problema de valor inicial en el contexto difuso: $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, donde $f : T \times X \to X$ es una aplicación con valores siendo conjuntos difusos, T es un intervalo de tiempo, X es una clase de conjuntos difusos, $x_0 \in X$ y $t_0 \in T$. Se considera la derivada $x'(t)$ como una generalización de la derivada de Hukuhara.

Palabras claves: Aplicaciones difusas, diferenciabilidad difusa, derivada de Hukuhara generalizada, ecuaciones diferenciales difusas, problema de Cauchy difuso.

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1. Introduction

Theory of fuzzy differential equations is a useful tool for modeling dynamical systems under possible uncertainty [19]. Fuzzy differential equations have been able to solve some disadvantages presented in the ordinary case. In particular, first order fuzzy differential equations appear in varied real problems, as for instance, quantum optics, gravity, medicine, chaotic systems, engineering problems, population models, etc. In general terms, the Cauchy problem associated with a first order fuzzy differential equation can be expressed as

$$x'(t) = f(t, x(t)),\ x(t_0) = x_0, \tag{1}$$

where $f : T \times X \to X$ is a fuzzy-valued mapping, $x_0 \in X, t_0 \in T$, T is an time interval and X is a class of fuzzy sets. So, to deal with the problem (1), we need to establish the sense of the derivative $x'(t)$. Initially, **Puri** and **Ralescu** [12] developed the concept of Hukuhara differentiability for fuzzy valued mappings (c.f. Definition 3.5). Under this setting, several results of existence and uniqueness of solutions of fuzzy differential equations (1) have been obtained (see for instance [7, 10, 11, 15, 16] and some references therein). However, this approach has the disadvantage that, in some cases, the support of the solutions have an increasing length as time t increases, which shows that this interpretation is not a good generalization of the corresponding crisp case (see [1]). In order to solve this difficultly, some approaches have been proposed. A first alternative is to replace the fuzzy differential equation in (1) by a family of differential inclusions (see [4, 5]). However, the approach of differential inclusions does not take into account the kind of fuzzy derivative of fuzzy-valued mapping. **Bede** and **Gal** [1] gave another possibility to solve this shortcoming by introducing a more general definition of derivative for fuzzy mappings, which allows to define the derivative for a larger class of fuzzy functions. This generalization of the Hukuhara derivative is obtained by considering the fuzzy lateral Hukuhara derivatives (c.f. Definition 3.6 below). Also, by interpreting the derivative $x'(t)$ in the generalized Hukuhara sense, some results of existence of solutions for the initial value fuzzy problem (1) were obtained in [1, 3]. Recently, in [8], by utilizing the generalized differentiability, the authors investigate the problem of finding new solutions for a second order fuzzy differential equation. We also refer the work [20], which presents recent results related to the global existence of solutions for fuzzy second-order differential equations under generalized H-differentiability. On the other hand, in [17] the authors investigate the first order linear fuzzy differential dynamical systems with fuzzy matrices. Especially, the authors discuss some properties of the 2D dynamical systems and describe their phase portraits. In this paper we prove the results of existence and uniqueness of solutions of the fuzzy initial value problem (1) developed in [1, 3, 7, 11, 15] (and some references therein), but assuming a more general definition of the derivative $x'(t)$. Indeed, we consider the derivative $x'(t)$ as being the generalized Hukuhara derivative of the set-valued mapping x_α defined by the α-levels of the fuzzy set $x(t)$ (c.f. Definition 3.9). This definition of differentiability will be called α-differentiability and it generalizes the notion of differentiability used in [1, 2, 3, 7, 11, 15, 16] and some references therein. Some properties related with the differential calculus taking into account the notion of α-differentiability are showed. In order to establish our existence result (c.f. Theorem 4.2) we prove an equivalence between the differential problem (1) and an integral formulation (c.f. Theorem 4.1). As our approach is based on the analysis of the set-valued mappings defined by the α-levels of the respective fuzzy-valued mappings, we assume a more general continuity condition on f; indeed we assume that f is a α-continuous function with respect to the Hausdorff metric d (c.f. Definition 4.1).

The outline of this paper is the following: in Section 2 we recall some preliminaries about the general theory of fuzzy sets. In Section 3 we introduce the definition of α-differentiability and give some results concerning the differential calculus. Finally, in Section 4, we analyze the Cauchy problem of first order fuzzy differential equations.

2. Preliminaries

Let $\mathcal{K}^n$ be the collection of all nonempty-convex-compact subsets of $\mathbb{R}^n$. If $A, B \in \mathcal{K}^n$ and $\lambda \in \mathbb{R}$, then the addition and the scalar multiplication in $\mathcal{K}^n$ are defined as:

$$A + B = \{a + b \mid a \in A,\ b \in B\}, \lambda A = \{\lambda a \mid a \in A\}. \tag{2}$$

In [14] was proved that $\mathcal{K}^n$ is a commutative semigroup under the addition, which verifies the cancellation law. Furthermore, can be proved that $\alpha(A+B) = \alpha A + \alpha B$, $\alpha(\beta A) = (\alpha\beta)A$ and $1A = A$ for $\alpha, \beta \in \mathbb{R}$, $A, B \in \mathcal{K}^n$. Moreover, if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$.

The Hausdorff metric d on $\mathcal{K}^n$ is defined as:

$$d(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} ||a - b||, \sup_{b\in B}\inf_{a\in A} ||a - b||\right\},$$

where $A, B \in \mathcal{K}^n$ and $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^n$. The couple $(\mathcal{K}^n, d)$ is a complete metric space; moreover the metric d verifies the following properties: $(i.)$ $d(\lambda A, \lambda B) = |\lambda| d(A, B)$, $(ii.)$ $d(A + C, B + C) = d(A, B)$ and $(iii.)$ $d(A+B, C+D) \leq d(A, C) + d(B, D)$, for all $A, B, C, D \in \mathcal{K}^n$ and $\lambda \in \mathbb{R}$.

A fuzzy set u on $\mathbb{R}^n$ is defined as a mapping $u : \mathbb{R}^n \to [0, 1]$. For $0 < \alpha \leq 1$, the class $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$ denotes the α-level of the fuzzy set u. For $\alpha = 0$, the support of u is defined as the set $[u]^0 = \text{supp}(u) = \overline{\{x \in \mathbb{R}^n \mid u(x) > 0\}}$.

Let $\mathcal{F}^n$ be the class of fuzzy sets $u : \mathbb{R}^n \to [0, 1]$ such that u satisfies:

(1) u is normal, that is, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,
(2) u is fuzzy convex, that is, $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,
(3) u is upper semicontinuous,
(4) $[u]^0 = \overline{\{x \in \mathbb{R}^n \mid u(x) > 0\}}$ is compact.

Then, from the definition of $\mathcal{F}^n$, we have that $[u]^\alpha \in \mathcal{K}^n$ for all $0 \leq \alpha \leq 1$. $\mathcal{F}^1$ is often called the class of fuzzy numbers. Real numbers $\mathbb{R}$ can be embedded in $\mathcal{F}^1$ by using the application $a \in \mathbb{R} \hookrightarrow \chi_{\{a\}} \in \mathcal{F}^1$. In general, $\mathcal{K}^n$ can be embedded in $\mathcal{F}^n$. We recall that if u, v are two fuzzy sets, then $u = v$ if and only if $[u]^\alpha = [v]^\alpha$, for all $\alpha \in [0, 1]$.

The following representation result is known as Negoita-Ralescu Theorem.

Theorem 2.1 ([9]). *If $u \in \mathcal{F}^n$, then*

(i) $[u]^\alpha \in \mathcal{K}^n$ *for all* $\alpha \in [0, 1]$,
(ii) $[u]^1 \subseteq [u]^\beta \subseteq [u]^\alpha \subseteq [u]^0$ *for all* $0 \leq \alpha \leq \beta \leq 1$,
(iii) *If* $\{\alpha_n\} \subset [0, 1]$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then* $[u]^\alpha = \bigcap_{n=1}^{\infty} [u]^{\alpha_n}$.

Conversely, if $\{N_\alpha \mid \alpha \in [0, 1]\}$ is a family of subsets of $\mathbb{R}^n$ satisfying $(i) - (iii)$, then there exists $u \in \mathcal{F}^n$ such that $[u]^\alpha = N_\alpha$, for all $\alpha \in (0, 1]$, and $[u]^0 = \overline{\bigcup_{0<\alpha\leq 1} N_\alpha} \subseteq N_0$.

According to the Zadeh Extension Principle [18], the operations of addition and scalar multiplication in $\mathcal{F}^n$ are defined as:

$$\begin{aligned} (u + v)(x) &= \sup_{y+z=x} \min\{u(y), v(z)\}, \\ (\lambda u)(x) &= \begin{cases} u(\frac{x}{\lambda}), & \lambda \neq 0, \\ \chi_{\{0\}}(x), & \lambda = 0, \end{cases} \end{aligned} \tag{3}$$

where $\lambda \in \mathbb{R}$, $x, y, z \in \mathbb{R}^n$ and $\chi_{\{0\}}$ is the characteristic function of $0 \in \mathbb{R}^n$. By the Zadeh Extension Principle and Theorem 2.1, the following relations hold:

$$\begin{aligned} [u + v]^\alpha &= [u]^\alpha + [v]^\alpha, \\ [\lambda u]^\alpha &= \lambda [u]^\alpha, \forall u, v \in \mathcal{F}^n, \forall \alpha \in [0, 1]. \end{aligned} \tag{4}$$

The Hausdorff metric d in $\mathcal{K}^n$ can be extended to $\mathcal{F}^n$ defining the distance

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]^\alpha, [v]^\alpha), \qquad \forall u, v \in \mathcal{F}^n.$$

The couple $(\mathcal{F}^n, D)$ is a complete metric space [13]. The metric D verifies the following properties: $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ and $D(u + w, v + w) = D(u, v)$, for all $u, v \in \mathcal{F}^n$, $\lambda \in \mathbb{R}$.

3. α-continuity and α-differentiability

We start by recalling some properties related to the measurability and the integrability of fuzzy set-valued mappings (c.f. [7]). Let $T = [a, b] \subset \mathbb{R}$ and consider $\mathcal{K}^n$ endowed with the Hausdorff metric d. We recall that a fuzzy-valued mapping $F : T \to \mathcal{F}^n$ is said to be strongly measurable, if for each $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : T \to \mathcal{K}^n$ given by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable. On the other hand, an application $F : T \to \mathcal{F}^n$ is called integrally bounded, if there exists a real-value integrable function g such that for each $y \in F_0(t)$ it holds $\|y\| \leq g(t)$. Now, we recall the integral of a fuzzy valued mapping.

Definition 3.1 ([7]). Let $T = [a, b] \subset \mathbb{R}$ and $F : T \to \mathcal{F}^n$. The integral $\int_a^b F(t)dt$ is defined levelwise by

$$\begin{aligned} &\left[\int_a^b F(t)dt\right]^\alpha = \int_a^b F_\alpha(t)dt \\ &= \left[\int_a^b f(t)dt \mid f : T \to \mathbb{R}^n \text{ is a measurable selection for } F_\alpha\right], \end{aligned} \tag{5}$$

for all $0 < \alpha \leq 1$. A strongly measurable and integrally bounded mapping $F : T \to \mathcal{F}^n$ is called to be integrable over the interval T if $\int_a^b F(t)dt$ belongs to $\mathcal{F}^n$.

Proposition 3.1 ([7]). *If $F : T \to \mathcal{F}^n$ is a strongly measurable and integrally bounded mapping, then F is integrable.*

Proposition 3.2 ([7]). *Let $T = [a, b] \subset \mathbb{R}$ and $c \in T$. If $F, G : T \to \mathcal{F}^n$ are integrable and $\lambda \in \mathbb{R}$, then*

(i) $\int_a^b F(t)dt = \int_a^c F(t)dt + \int_c^b F(t)dt$,
(ii) $\int_a^b (\lambda F(t) + G(t))dt = \lambda \int_a^b F(t)dt + \int_a^b G(t)dt$,
(iii) $D(F, G)$ *is integrable,*
(iv) $D(\int_a^b F(t)dt, \int_a^b G(t)dt) \leq \int_a^b D(F, G)(t)dt$.

Remark 3.1 ([7]). If $F : T \to \mathcal{F}^1$ is integrable such that $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$ for all $\alpha \in [0, 1]$, then $\int_T F$ is obtained by integrating the α-level curves, that is,

$$\left[\int_T F\right]^\alpha = \left[\int_T f_\alpha, \int_T g_\alpha\right].$$

Remark 3.2 ([7]). Let $A \in \mathcal{F}^n$ and consider the fuzzy-valued mapping $F : T \to \mathcal{F}^n$ given by $F(s) = A$ for all $0 \leq s \leq t$. Then

$$\int_0^t F = tA.$$

Definition 3.2. Let $T = [a, b] \subset \mathbb{R}$. A fuzzy-valued mapping $F : T \to \mathcal{F}^n$ is continuous at a point $t_0 \in T$, if given $\epsilon > 0$, there exists $\delta > 0$ such that $D(F(t), F(t_0)) < \epsilon$, for $t \in T$ satisfying $|t - t_0| < \delta$.

Throughout this paper we will consider the following definition of continuity for fuzzy-valued mappings, which generalizes Definition 3.2.

Definition 3.3 ([11, 15]). Let $T = [a, b] \subset \mathbb{R}$. A mapping $F : T \to \mathcal{F}^n$ is said α-continuous at $t_0 \in T$, if the set-valued mappings $F_\alpha : T \to \mathcal{K}^n$ defined by $F_\alpha(t) = [F(t)]^\alpha$, $\alpha \in [0, 1]$, are continuous at $t = t_0$ with respect to the Hausdorff metric d, that is, fixed α and given $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$d([F(t)]^\alpha, [F(t_0)]^\alpha) < \epsilon,$$

for all t at T with $|t - t_0| < \delta$. If F is α-continuous for all $t \in T$, we simply say that F is α-continuous.

Proposition 3.3 ([15]). *Let $T = [a, b] \subset \mathbb{R}$. If $F : T \to \mathcal{F}^n$ is α-continuous, then F is integrable.*

Now we introduce the concept of α-differentiability which will be used throughout this paper.

The *Hukuhara difference* (H-difference) $A \ominus B$ for $A, B \in \mathcal{K}^n$ (if it exists), is defined to be the set $C \in \mathcal{K}^n$ such that $A = B + C$. In general $A \ominus B \neq A + (-)B = A - B$. Based on the definition of H-difference, in [6], Hukuhara gave the following definition of Hukuhara differentiability for set-valued mappings.

Definition 3.4 ([6]). Let $T = [a, b] \subset \mathbb{R}$ and $G : T \to \mathcal{K}^n$. G is Hukuhara differentiable (H-differentiable) at $t_0 \in T$ if for $h > 0$ small enough, the differences $G(t_0 + h) \ominus G(t_0)$, $G(t_0) \ominus G(t_0 - h)$ exist, and there exists $G'(t_0) \in \mathcal{K}^n$ such that

$$\begin{aligned}\lim_{h\to 0^+} \frac{G(t_0 + h) \ominus G(t_0)}{h} &= \lim_{h\to 0^+} \frac{G(t_0) \ominus G(t_0 - h)}{h} \\ &= G'(t_0),\end{aligned}$$

where the limits are taken in the metric space $(\mathcal{K}^n, d)$. At the end points of the interval T one considers only the one-side derivatives.

Based on Definition 3.4, **Puri** and **Ralescu** [12] extended the notion of H-derivative of a fuzzy-valued mapping. In fact, for $u, v \in \mathcal{F}^n$, an element $w \in \mathcal{F}^n$ (if it exists) such that $u = v + w$, is called the H-difference of u and v and it is denoted by $u \ominus v$. Then the following definition is established.

Definition 3.5 ([12]). Let $T = [a, b] \subset \mathbb{R}$ and consider a fuzzy mapping $F : T \to \mathcal{F}^n$. F is said H-differentiable at a point $t_0 \in T$ if for $h > 0$ small enough, the differences $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ exist, and there exists an element $F'(t_0) \in \mathcal{F}^n$ such that

$$\begin{aligned}&\lim_{h\to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \\ &\qquad = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0),\end{aligned} \tag{6}$$

where the limits are taken in the metric space $(\mathcal{F}^n, D)$. At the end points of the interval T one considers only the one-side derivatives.

As pointed out in [1], the definition of H-derivative of a fuzzy-valued mapping is very restrictive. Indeed, if we consider a fuzzy number c and $g : [a, b] \to \mathbb{R}$ a real-valued function, differentiable at $t_0 \in (a, b)$ with $g'(t_0) \leq 0$, then the fuzzy-valued mapping $f(x) = cg(x)$ is not H-differentiable at t_0. To solve this shortcoming, the authors of [1] introduced the notion of generalized derivative by taking into account the lateral types of H-derivatives, as follows.

Definition 3.6 ([3, 1]). Let $T = [a, b] \subset \mathbb{R}$ and consider $F : T \to \mathcal{F}^n$. F is differentiable at $t_0 \in T$, in the generalized sense, if

(1) for $h > 0$ small enough, the differences $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ exist, and there exists $F'(t_0) \in \mathcal{F}^n$ such that

$$\lim_{h\to 0^+} \frac{F(t_0+h)\ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0)\ominus F(t_0-h)}{h} = F'(t_0), \tag{7}$$

or

(2) for $h < 0$ small enough, the differences $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ exist, and there exists $F'(t_0) \in \mathcal{F}^n$ such that

$$\lim_{h\to 0^-} \frac{F(t_0+h)\ominus F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0)\ominus F(t_0-h)}{h} = F'(t_0), \tag{8}$$

where the limits are taken in the metric space $(\mathcal{F}^n, D)$. At the end points of the interval T one considers only the one-side derivatives.

Remark 3.3. In [3] the authors considered the two cases 1 and 2 of Definition 3.6. In [1], in addition to cases 1 and 2 of Definition 3.6, the authors considered other two cases but in the other two cases, the derivative is trivial because it is reduced to a crisp set (c.f. Remark 2 in [3]).

An analogous definition holds for the case of set-valued mappings.

Definition 3.7. Let $T = [a, b] \subset \mathbb{R}$ and consider $G : T \to \mathcal{K}^n$. G is differentiable at $t_0 \in T$, in the generalized sense, if

(1) for $h > 0$ small enough, the differences $G(t_0 + h) \ominus G(t_0)$, $G(t_0) \ominus G(t_0 - h)$ exist, and there exists $G'(t_0) \in \mathcal{K}^n$ such that

$$\lim_{h\to 0^+} \frac{G(t_0+h)\ominus G(t_0)}{h} = \lim_{h\to 0^+} \frac{G(t_0)\ominus G(t_0-h)}{h} = G'(t_0), \tag{9}$$

or

(2) for $h < 0$ small enough, the differences $G(t_0 + h) \ominus G(t_0)$, $G(t_0) \ominus G(t_0 - h)$ exist, and there exists $G'(t_0) \in \mathcal{K}^n$ such that

$$\lim_{h\to 0^-} \frac{G(t_0+h)\ominus G(t_0)}{h} = \lim_{h\to 0^-} \frac{G(t_0)\ominus G(t_0-h)}{h} = G'(t_0), \tag{10}$$

where the limits are taken in the metric space $(\mathcal{K}^n, d)$. At the ends points of the interval T one considers only the one-side derivatives. If G verifies (9) (respectively (10)) we say that G is differentiable in the generalized sense, in the first form, (respectively, G is differentiable in the generalized sense, in the second form).

On the other hand, in [11, 15], following notion of derivative was introduced, more general than Definition 3.5, by considering the Hukuhara-derivative of the respective set-valued mappings defined through the α-levels.

Definition 3.8 ([11, 15]). Let $T = [a, b] \subset \mathbb{R}$. A mapping $F : T \to \mathcal{F}^n$ is differentiable at the point $t_0 \in T$, if for every $\alpha \in [0, 1]$, the set-valued mapping, $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at the point t_0 according to the Definition 3.4 and the family $\{F'_\alpha(t) \mid \alpha \in [0, 1]\}$ define a fuzzy set $F'(t_0) \in \mathcal{F}^n$.

In this paper we enlarge the class of generalized differentiable fuzzy-valued mappings given in Definitions 3.6 and 3.8 (consequently Definition 3.5), by considering the lateral type of Hukuhara-derivatives of the respective set-valued mappings defined through the α-levels; more exactly, we have the following definition.

Definition 3.9. Let $T = [a, b] \subset \mathbb{R}$ and consider $F : T \to \mathcal{F}^n$. We say that F is α-differentiable at $t_0 \in T$, if for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : T \to \mathcal{K}^n$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is differentiable at the point t_0 in the generalized sense, and additionally, the family $\{F'_\alpha(t_0) : \alpha \in [0, 1]\}$ defines a fuzzy set $F'(t_0) \in \mathcal{F}^n$. If F is α-differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the derivative of F at t_0. If for each $\alpha \in [0, 1]$ the mapping F_α is differentiable in the generalized sense according (9) (respectively (10)) in Definition 3.7, we say that F is α-differentiable in the first form (respectively, second form).

Remark 3.4. The Definition 3.9 is a generalization of the notion of differentiability introduced by Seikkala [16] for studying fuzzy process. From Definition 3.9 it follows that if F is differentiable in the generalized sense according Definition 3.6, then F is α-differentiable (differentiable in the sense of Definition 3.9). The converse result is not true.

Example 3.0.1. The existence of Hukuhara differences $[u]^\alpha \ominus [v]^\alpha, \alpha \in [0, 1]$, do not imply the existence of $u \ominus v$. In fact, consider for instance $u : \mathbb{R} \to [0, 1]$ and

$v : \mathbb{R} \to [0,1]$, the fuzzy sets defined by

$$u(t) = \begin{cases} 1, & \text{if } t \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

$$v(t) = \begin{cases} -t+1, & \text{if } t \in [0,1] \\ t+1, & \text{if } t \in [-1,0] \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the Hukuhara differences between the α-levels of u and v exist, that is,

$$[u]^\alpha \ominus [v]^\alpha = [-1,1] \ominus [-1+\alpha, 1-\alpha] = [-\alpha, \alpha],$$

however the Hukuhara difference $u \ominus v$ does not exist, since the family $[-\alpha, \alpha]$, with $\alpha \in [0,1]$, does not define a fuzzy set (see Theorem 2.1).

The following example shows that the α-differentiability does not imply the differentiability in the generalized sense according to the Definition 3.6.

Example 3.0.2. Let $F : [0,2] \to \mathcal{F}^1$ a fuzzy number defined as

$$F(t)(x) = \chi_{[-1,1]}(x), \qquad \text{if} \quad t \in [1,2],\ x \in \mathbb{R},$$

and

$$F(t)(x) = \begin{cases} \frac{1}{(1-t)^2}(x+1), x \in [-1, -1+(1-t)^2], \\ \frac{-1}{(1-t)^2}(x-1), x \in [1, 1-(1-t)^2], \\ 0, x \notin (-1,1), \\ 1, x \in [-1+(1-t)^2, 1-(1-t)^2], \end{cases}$$

if $t \in [0,1)$. It is clear that the differences $F(1) \ominus F(1-h), h > 0$, do not exist (see Example 3.0.1). Hence, F is not differentiable in the generalized sense according to the Definition 3.6 at $t_0 = 1$. On the other hand, the family of set-valued mappings F_α (associated to F) are given by

$$F_\alpha(t) = \begin{cases} [-1+\alpha(1-t)^2, 1-\alpha(1-t)^2], & t \in [0,1), \\ [-1,1], & t \in [1,2]. \end{cases}$$

Now we calculate the α-derivative of F in the first form at $t_0 = 1$. It is clear that the differences of the α-levels exist. Then

$$\begin{aligned} &\frac{F_\alpha(1) \ominus F_\alpha(1-h)}{h} \\ &= \frac{[-1,1] \ominus [-1+\alpha(1-(1-h))^2, 1-\alpha(1-(1-h))^2]}{h} \\ &= \frac{[-1,1] \ominus [-1+\alpha h^2, 1-\alpha h^2]}{h} = \frac{[-\alpha h^2, \alpha h^2]}{h} \\ &= h[-\alpha, \alpha] \stackrel{h \to 0^+}{\longrightarrow} \{0\}. \end{aligned}$$

Moreover,

$$\frac{F_\alpha(1+h) \ominus F_\alpha(1)}{h} = \frac{[-1,1] \ominus [-1,1]}{h} = \{0\} \stackrel{h \to 0^+}{\longrightarrow} \{0\}.$$

Then $F'_\alpha(1) = \{0\}$ for all $\alpha \in [0,1]$. Hence F is α-differentiable at $t_0 = 1$ and its α-derivative is given by $F'(1) = \chi_{\{0\}}$.

Theorem 3.1. *Let $T = [a,b] \subset \mathbb{R}$, $F : T \to \mathcal{F}^1$ and denote by $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$, the respective set-valued mapping defined by the α-levels. Then*

(i) *If F is α-differentiable at t_0 in the first form, then f_α and g_α are differentiable and $[F'(t_0)]^\alpha = [f'_\alpha(t_0), g'_\alpha(t_0)]$.*
(ii) *If F is α-differentiable at t_0 in the second form, then f_α, g_α are differentiable and $[F'(t_0)]^\alpha = [g'_\alpha(t_0), f'_\alpha(t_0)]$.*

Proof: The part (3.1) was proved in [15]. So, we will prove the part (3.1). By hypothesis, for $h < 0$ small enough and given $\alpha \in [0,1]$ fixed but arbitrary, the differences $F_\alpha(t_0+h) \ominus F_\alpha(t_0)$ and $F_\alpha(t_0) \ominus F_\alpha(t_0-h)$ exist. Firstly note that by the definition of Hukuhara difference we get

$$\begin{aligned} &F_\alpha(t_0+h) \ominus F_\alpha(t_0) \\ &\quad = [f_\alpha(t_0+h), g_\alpha(t_0+h)] \ominus [f_\alpha(t_0), g_\alpha(t_0)] \\ &\quad = [f_\alpha(t_0+h) - f_\alpha(t_0), g_\alpha(t_0+h) - g_\alpha(t_0)]. \end{aligned}$$

Then, as $h < 0$, multiplying by $1/h$ we obtain

$$\begin{aligned} &\frac{F_\alpha(t_0+h) \ominus F_\alpha(t_0)}{h} \\ &\quad = \frac{1}{h}[f_\alpha(t_0+h) - f_\alpha(t_0), g_\alpha(t_0+h) - g_\alpha(t_0)] \\ &\quad = \Big[\frac{g_\alpha(t_0+h) - g_\alpha(t_0)}{h}, \frac{f_\alpha(t_0+h) - f_\alpha(t_0)}{h}\Big]. \end{aligned}$$

Taking the limit when $h \to 0^-$ one can ensure the existence of $g'_\alpha(t_0)$, $f'_\alpha(t_0)$, and for all $\alpha \in [0,1]$ it holds

$$\begin{aligned} F'_\alpha(t) &= \lim_{h \to 0^-} \frac{F_\alpha(t_0+h) \ominus F_\alpha(t_0)}{h} \\ &= \lim_{h \to 0^-} \Big[\frac{g_\alpha(t_0+h) - g_\alpha(t_0)}{h}, \frac{f_\alpha(t_0+h) - f_\alpha(t_0)}{h}\Big] \\ &= [g'_\alpha(t_0), f'_\alpha(t_0)]. \end{aligned}$$

Analogously we obtain

$$\begin{aligned} &\frac{F_\alpha(t_0) \ominus F_\alpha(t_0-h)}{h} \\ &\quad = \Big[\frac{g_\alpha(t_0) - g_\alpha(t_0-h)}{h}, \frac{f_\alpha(t_0) - f_\alpha(t_0-h)}{h}\Big], \end{aligned}$$

and hence

$$F'_\alpha(t) = \lim_{h\to 0^-} \frac{F_\alpha(t_0) \ominus F_\alpha(t_0 - h)}{h}$$
$$= \lim_{h\to 0^-} \Big[\frac{g_\alpha(t_0) - g_\alpha(t_0-h)}{h}, \frac{f_\alpha(t_0) - f_\alpha(t_0-h)}{h}\Big]$$
$$= [g'_\alpha(t_0), f'_\alpha(t_0)].$$

Theorem 3.2. *Let $T = [a,b] \subset \mathbb{R}$. If $F : T \to \mathcal{F}^n$ is α-differentiable, then it is α-continuous.*

Proof: In [15] was proved that if F is α-differentiable in the first form, then it is α-continuous. In our case, let $t, t+h \in T$ with $h < 0$, h small enough and $\alpha \in [0,1]$. By hypothesis, the differences $F_\alpha(t+h) \ominus F_\alpha(t)$ exist. Now, by using the properties of the metric d we get

$$d(F_\alpha(t+h), F_\alpha(t))$$
$$= d(F_\alpha(t) + F_\alpha(t+h) \ominus F_\alpha(t), F_\alpha(t))$$
$$= d(F_\alpha(t+h) \ominus F_\alpha(t), \{0\})$$
$$= |h| d\left(\frac{F_\alpha(t+h) \ominus F_\alpha(t)}{h}, \{0\}\right)$$
$$\le |h| d\left(\frac{F_\alpha(t+h) \ominus F_\alpha(t)}{h}, F'_\alpha(t)\right) + |h| d\left(F'_\alpha(t), \{0\}\right).$$

As F is α-differentiable in the second form, then if $h \to 0^-$, the right-hand side of last inequality tends to zero, and hence F is left α-continuous. In an analogous way, by working with the difference $F_\alpha(t) \ominus F_\alpha(t-h)$, we can prove that F is right α-continuous.

Theorem 3.3. *Let $T = [a,b] \subset \mathbb{R}$. If $F, G : T \to \mathcal{F}^n$ are α-differentiable at the point $t \in T$ and $\lambda \in \mathbb{R}$, then $(F+G)'(t) = F'(t) + G'(t)$ and $(\lambda F)'(t) = \lambda F'(t)$.*

Proof: The proof follows by using Lemma 3 in [14] and basic properties of the metric d.

Theorem 3.4. *Let $T = [a,b] \subset \mathbb{R}$. If $F : T = [a,b] \to \mathcal{F}^n$ is α-continuous, then*

(a) $G(t) = \int_a^t F$ *is α-differentiable in the first form and $G'(t) = F(t)$ for all $t \in T$, and,*
(b) $H(t) = \int_t^b F$ *is α-differentiable in the second form and $H'(t) = -F(t)$ for all $t \in T$.*

Proof: If F is α-continuous, by Theorem 3.3 we conclude that F_α is integrable; hence, the fuzzy-valued mappings G and H are well-defined. The proof in the case (3.4) was given in [15]. We will prove the case (3.4). Let $h < 0$ small enough such that $a \le t+h < t \le b$. By Theorem 3.2, item (3.2), it holds

$$\int_{t+h}^t F_\alpha + \int_t^b F_\alpha = \int_{t+h}^b F_\alpha,$$

that is,

$$\int_{t+h}^t F_\alpha + H_\alpha(t) = H_\alpha(t+h),$$

or equivalently

$$H_\alpha(t+h) \ominus H_\alpha(t) = \int_{t+h}^t F_\alpha.$$

Let $\epsilon > 0$. By properties of metric d, Remark 3.2 and the α-continuity of F_α, we have

$$d\left(\frac{H_\alpha(t+h) \ominus H_\alpha(t)}{h}, -F_\alpha(t)\right)$$
$$= \frac{1}{|h|} d\left(H_\alpha(t+h) \ominus H_\alpha(t), -hF_\alpha(t)\right)$$
$$= \frac{1}{|h|} d\left(\int_{t+h}^t F_\alpha(s)ds, \int_{t+h}^t F_\alpha(t)ds\right)$$
$$\le \frac{1}{|h|}\int_{t+h}^t d(F_\alpha(s), F_\alpha(t))ds \le \epsilon,$$

for $h < 0$ small enough. Consequently,

$$\lim_{h\to 0^-} \frac{H_\alpha(t+h) \ominus H_\alpha(t)}{h} = -F_\alpha(t).$$

In an analogous way we get

$$\lim_{h\to 0^-} \frac{H_\alpha(t) \ominus H_\alpha(t-h)}{h} = -F_\alpha(t).$$

Theorem 3.5. *Let $T = [a,b] \subset \mathbb{R}$, $F : T \to \mathcal{F}^1$ and denote by $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$, the respective set-valued mapping defined through the α-levels of F. If F is α-differentiable in the second form on the interval T, with F' such that $[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]$ verifies that $g'_\alpha(t), f'_\alpha(t)$ are continuous functions on T, then for each $s \in T$ we have*

$$F(s) = F(a) \ominus (-1)\int_a^s F'(t)\,dt. \tag{11}$$

Remark 3.5. In Theorem 3.5, in case when F is α-differentiable in the first form on the interval T, we have that $F(s) = F(a) + \int_a^s F'(t)\,dt$. (c.f [11, 15]).

Proof: As F is α-differentiable in the second form, from Theorem 3.1, part (3.1), we have that $[F'(t)]^\alpha =$

$[g'_\alpha(t), f'_\alpha(t)]$. Now, by using Remark 3.1, for each $\alpha \in [0,1]$, we get

$$\left[\int_a^s F'(t)dt\right]^\alpha = \int_a^s [F'(t)]^\alpha dt = \left[\int_a^s g'_\alpha(t)dt, \int_a^s f'_\alpha(t)dt\right] = [g_\alpha(s) - g_\alpha(a), f_\alpha(s) - f_\alpha(a)].$$

Consequently,

$$(-1)\int_a^s [F'(t)]^\alpha\, dt = [f_\alpha(a) - f_\alpha(s), g_\alpha(a) - g_\alpha(s)] = [f_\alpha(a), g_\alpha(a)] \ominus [f_\alpha(s), g_\alpha(s)] = F_\alpha(a) \ominus F_\alpha(s) = [F(a)]^\alpha \ominus [F(s)]^\alpha.$$

Then, for all $\alpha \in [0,1]$, we obtain

$$[F(a)]^\alpha = [F(s)]^\alpha + (-1)\int_a^s [F'(t)]^\alpha\, dt,$$

that is,

$$[F(s)]^\alpha = [F(a)]^\alpha \ominus (-1)\int_a^s [F'(t)]^\alpha\, dt.$$

Therefore (11) is proved.

4. Cauchy problem of fuzzy differential equations

Definition 4.1 ([11, 15]). Let $T = [a,b] \subset \mathbb{R}$. A function $f : T \times \mathcal{F}^n \to \mathcal{F}^n$ is called α-continuous at a point $(t_0, x_0) \in T \times \mathcal{F}^n$ provided for any fixed $\alpha \in [0,1]$ and for any $\epsilon > 0$, there exists $\delta(\alpha, \epsilon) > 0$ such that

$$d([f(t,x)]^\alpha, [f(t_0,x_0)]^\alpha) < \epsilon, \tag{12}$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and

$$d([x]^\alpha, [x_0]^\alpha) < \delta(\alpha, \epsilon),\ t \in T, x \in \mathcal{F}^n.$$

The aim of this section is to analyze the existence and uniqueness of solutions of the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{13}$$

where $x_0 \in \mathcal{F}^1$, $f : T \times \mathcal{F}^1 \to \mathcal{F}^1$ is α-continuous and x' denotes the α-derivative of the mapping x. The following theorem gives an equivalence between the fuzzy differential equation and an integral formulation.

Theorem 4.1. *Let $f : T \times \mathcal{F}^1 \to \mathcal{F}^1$ α-continuous and $x_0 \in \mathcal{F}^1$. A mapping $x : T \to \mathcal{F}^1$ is a solution of (13) if and only if x is α-continuous and verifies the integral equation*

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds,\ \forall t \in T, \tag{14}$$

or

$$x(t) = x_0 \ominus (-1)\int_{t_0}^t f(s, x(s))ds,\ \forall t \in T, \tag{15}$$

depending on the α-differentiability considered, first or second form, respectively.

Proof: Using the arguments of [7], in [15] was proved that if the α-differentiability is considered in the first form, then a function $x : T \to \mathcal{F}^1$ is a solution of (13) if and only if x is α-continuous and verifies the integral equation (14). We will prove the second equivalence. Firstly, we note that if x is a solution of (13) with the α-derivative x' being considered in the second form, then from Theorem 3.2 it follows that x is α-continuous. Moreover, Theorem 3.5 for all $t \in T$, implies that

$$x(t) = x_0 \ominus (-1)\int_{t_0}^t f(s, x(s))ds = x_0 \ominus (-1)\int_{t_0}^t x'(s)ds.$$

On the other hand, by Theorem 3.3, if f is α-continuous then f is integrable; hence, the integral in (15) has meaning. Moreover, if x is α-continuous and verifies the integral equation (15), then by using Remark 3.1, the α-levels of x (denoted by $[x(t)]^\alpha = [x^1_\alpha(t), x^2_\alpha(t)]$) verify

$$[x(t)]^\alpha = [x^1_\alpha(t), x^2_\alpha(t)] = [x_0]^\alpha \ominus (-1)\left[\int_{t_0}^t f(s, x(s))ds\right]^\alpha = [x^1_\alpha(t_0), x^2_\alpha(t_0)] \ominus (-1)\int_{t_0}^t [f(s,x(s))]^\alpha ds$$
$$= [x^1_\alpha(t_0), x^2_\alpha(t_0)] \ominus (-1)\left[\int_{t_0}^t f^1_\alpha(s, x(s))ds, \int_{t_0}^t f^2_\alpha(s, x(s))ds\right]$$
$$= [x^1_\alpha(t_0), x^2_\alpha(t_0)] \ominus \left[-\int_{t_0}^t f^2_\alpha(s, x(s))ds, -\int_{t_0}^t f^1_\alpha(s, x(s))ds\right],$$

where f^1_α, f^2_α are defined such that

$$[f(s,x(s))]^\alpha = [f^1_\alpha(s,x(s)), f^2_\alpha(s,x(s))].$$

Consequently

$$[x^1_\alpha(t_0), x^2_\alpha(t_0)] = \left[-\int_{t_0}^t f^2_\alpha(s,x(s))ds, -\int_{t_0}^t f^1_\alpha(s,x(s))ds\right] + [x^1_\alpha(t), x^2_\alpha(t)],$$

and therefore

$$x_\alpha^1(t) = x_\alpha^1(t_0) + \int_{t_0}^t f_\alpha^2(s, x(s))ds,$$
$$x_\alpha^2(t) = x_\alpha^2(t_0) + \int_{t_0}^t f_\alpha^1(s, x(s))ds.$$

Thus,

$$(x_\alpha^1)'(t) = f_\alpha^2(t, x(t)), \qquad (x_\alpha^2)'(t) = f_\alpha^1(t, x(t)),$$

or equivalently,

$$\begin{aligned}[x'(t)]^\alpha &= [(x_\alpha^2)'(t), (x_\alpha^1)'(t)] \\ &= [f_\alpha^1(t, x(t)), f_\alpha^2(t, x(t))] \\ &= [f(t, x(t))]^\alpha, \ \forall \alpha \in [0, 1],\end{aligned}$$

which proves that x is a solution of the differential equation (13).

The following theorem gives us conditions on the existence (and, in a certain sense, the uniqueness) of solutions for the fuzzy initial value problem (13) when we consider the notion of α-differentiability.

Theorem 4.2. *Let us suppose that the following conditions hold*

1. *A mapping $f : R_0 \to \mathcal{F}^1$ is α-continuous, where $R_0 = \{t : |t - t_0| \leq \delta \leq a\} \times \{x \in \mathcal{F}^1 : D(x, x_0) \leq b\}$, $a > 0, b > 0$, and $x_0 \in \mathcal{F}^1$.*
2. *There exists $K > 0$ such that for all $(t, x), (t, y) \in R_0$, $d([f(t, x)]^\alpha, [f(t, y)]^\alpha) \leq K d([x]^\alpha, [y]^\alpha)$, for all $\alpha \in [0, 1]$.*
3. *There exists $q > 0$ such that for any t satisfying $|t - t_0| \leq q$, the sequence $\tilde{x}_n(t) = x_0 \ominus (-1) \int_{t_0}^t f(s, \tilde{x}_{n-1}(s))ds$ are defined for all $n \in \mathbb{N}$.*

Then the fuzzy initial value problem (13) has two (unique) solutions $x, \tilde{x}$, with x being α-differentiable in the first form, $\tilde{x}$ being α-differentiable in the second form, and defined on the interval

$$|t - t_0| \leq \delta = min\left\{a, \frac{b}{M}, q\right\}, \tag{16}$$

where $M = D(f(t, x), \hat{0})$ for any $(t, x) \in R_0$, and $\hat{0} \in \mathcal{F}^1$ is the fuzzy number defined by

$$\hat{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & otherwise. \end{cases}$$

Moreover

$$D(x_n, x) \to 0, D(\tilde{x}_n, \tilde{x}) \to 0, \ on \ |t - t_0| \leq \delta, \tag{17}$$

as $n \to \infty$, where $x_n, \tilde{x}_n$ are the respective successive approximations given by

$$x_n(t) = x_0 + \int_{t_0}^t f(s, x_{n-1}(s))ds, \ n = 1, 2, \ldots \tag{18}$$

$$\tilde{x}_n(t) = x_0 \ominus (-1) \int_{t_0}^t f(s, \tilde{x}_{n-1}(s))ds, \ n = 1, 2, \ldots \tag{19}$$

Remark 4.1. The importance of this theorem is that it guarantees the existence (and uniqueness) for fuzzy initial value problem by assuming a more general notion of differentiability, as was developed in [1, 3, 7, 11, 15]. Moreover, this theorem provides conditions for the implementation of a numerical method in order to obtain such solutions.

Proof: For the case of α-differentiability in the first form, we obtain the existence of a unique solution x verifying $D(x_n, x) \to 0$, and x_n as in (18) (c.f. [11]). For the case of α-differentiability in the second form, if $t \in \{t : |t - t_0| \leq \delta \leq a\}$, then for $k = 1$

$$\tilde{x}_1(t) = x_0 \ominus (-1) \int_{t_0}^t f(s, x_0(s))ds,$$

and hence, from Theorem 4.1, the function $\tilde{x}_1$ is α-continuous on the interval $|t - t_0| \leq \delta$. Moreover, for all $\alpha \in [0, 1]$, by using (4) and properties of metric d given in Section 2, we have

$$\begin{aligned} &d([\tilde{x}_1(t)]^\alpha, [x_0]^\alpha) = \\ &\quad d\left([\tilde{x}_1]^\alpha, [\tilde{x}_1]^\alpha + (-1)\left[\int_{t_0}^t f(s, x_0(s))ds\right]^\alpha\right) = \\ &\qquad d\left(\left[\int_{t_0}^t f(s, x_0(s))ds\right]^\alpha, \{0\}\right). \end{aligned}$$

Consequently, taking the $\sup\limits_{0 \leq \alpha \leq 1}$ in the last equality we obtain

$$\begin{aligned} &D(\tilde{x}_1(t), x_0) = \\ &\quad D\left(\int_{t_0}^t f(s, x_0(s))ds, \hat{0}\right) \leq \\ &\quad \int_{t_0}^t D\left(f(s, x_0(s)), \hat{0}\right) ds \leq \\ &\qquad M|t - t_0| \leq M\delta \leq b, \end{aligned}$$

provided $|t - t_0| \leq \delta$, where $M = D(f(t, x), \widehat{0})$, for any $(t, x) \in R_0$. By an induction argument, assume that $\tilde{x}_{n-1}(t)$ is α-continuous on $|t - t_0| \leq \delta$ and $D(\tilde{x}_{n-1}(t), x_0) \leq M|t - t_0| \leq M\delta \leq b$, provided

$|t-t_0| \leq \delta$. Using (19) we have that $\tilde{x}_n(t)$ is α-continuous on $|t-t_0| \leq \delta$ and

$$D(\tilde{x}_n(t), x_0) \leq M|t-t_0| \leq M\delta \leq b.$$

Consequently we have that the set $\{\tilde{x}_n(t)\}_{n\geq 1}$ is a sequence of functions which are α-continuous on $|t-t_0| \leq \delta$ and $(t, \tilde{x}_n(t)) \in R_0$, $|t-t_0| \leq \delta$, $n = 1, 2, \ldots$

We will show that there exists $\tilde{x} : \{t : |t-t_0| \leq \delta \leq a\} \to \mathcal{F}^1$ such that $D(\tilde{x}_n(t), \tilde{x}(t)) \to 0$ uniformly on $|t-t_0| \leq \delta$, when $n \to \infty$. Note that by definition of

Hukuhara difference, we have

$$x_0 = (-1)\int_{t_0}^{t} f(s, x_0(s))ds + \tilde{x}_1(t),$$

$$x_0 = (-1)\int_{t_0}^{t} f(s, x_1(s))ds + \tilde{x}_2(t).$$

Then, by using the properties of the Hausdorff distance d (including the invariance with respect to translations), for any $\alpha \in [0,1]$ we obtain

$$\begin{aligned}
&d([\tilde{x}_2(t)]^\alpha, [\tilde{x}_1(t)]^\alpha) \leq \cdots \\
&\leq d\left([\tilde{x}_2(t)]^\alpha + \left[(-1)\int_{t_0}^{t} f(s,\tilde{x}_1(s))ds\right]^\alpha, [\tilde{x}_1(t)]^\alpha + \left[(-1)\int_{t_0}^{t} f(s,x_0(s))ds\right]^\alpha\right) \\
&\quad + d\Big([\tilde{x}_2(t)]^\alpha + \left[(-1)\int_{t_0}^{t} f(s,\tilde{x}_1(s))ds\right]^\alpha, [\tilde{x}_2(t)]^\alpha + \left[(-1)\int_{t_0}^{t} f(s,x_0(s))ds\right]^\alpha\Big) \\
&= d([\tilde{x}_0]^\alpha, [\tilde{x}_0]^\alpha) + d\left(\left[\int_{t_0}^{t} f(s,\tilde{x}_1(s))ds\right]^\alpha, \left[\int_{t_0}^{t} f(s,x_0(s))ds\right]^\alpha\right) \\
&\leq \int_{t_0}^{t} d([f(s,\tilde{x}_1(s))]^\alpha, [f(s,x_0(s))]^\alpha)ds \leq \int_{t_0}^{t} Kd([\tilde{x}_1(s)]^\alpha, [x_0(s)]^\alpha)ds,
\end{aligned}$$

and thus

$$\begin{aligned}
D(\tilde{x}_2(t), \tilde{x}_1(t)) &\leq K\int_{t_0}^{t} D(\tilde{x}_1(s), x_0(s))ds \\
&\leq MK\frac{|t-t_0|^2}{2!} \leq MK\frac{\delta^2}{2!}.
\end{aligned}$$

By an induction argument, assume that

$$D(\tilde{x}_n(t), \tilde{x}_{n-1}(t)) \leq MK^{n-1}\frac{|t-t_0|^n}{n!} \leq MK^{n-1}\frac{\delta^n}{n!}.$$

We want to prove that

$$\begin{aligned}
&D(\tilde{x}_{n+1}(t), \tilde{x}_n(t)) \\
&\qquad \leq MK^n\frac{|t-t_0|^{n+1}}{(n+1)!} \leq MK^n\frac{\delta^{n+1}}{(n+1)!}. \qquad (20)
\end{aligned}$$

In fact, as before, for all $\alpha \in [0,1]$ we obtain

$$\begin{aligned}
&d([\tilde{x}_{n+1}(t)]^\alpha, [\tilde{x}_n(t)]^\alpha) \\
&\leq d\left(\left[\int_{t_0}^{t} f(s,\tilde{x}_n(s))ds\right]^\alpha, \left[\int_{t_0}^{t} f(s,\tilde{x}_{n-1}(s))ds\right]^\alpha\right) \\
&\qquad \leq \int_{t_0}^{t} d([f(s,\tilde{x}_n(s))]^\alpha, [f(s,\tilde{x}_{n-1}(s))]^\alpha)ds \\
&\qquad \leq \int_{t_0}^{t} Kd([\tilde{x}_n(s)]^\alpha, [\tilde{x}_{n-1}(s)]^\alpha)ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&D(\tilde{x}_{n+1}(t), \tilde{x}_n(t)) \\
&\qquad \leq K\int_{t_0}^{t} D(\tilde{x}_n(t), \tilde{x}_{n-1}(t))ds \\
&\qquad \leq MK^n\int_{t_0}^{t} \frac{|s-t_0|^n}{n!} \leq MK^n\frac{\delta^{n+1}}{(n+1)!}.
\end{aligned}$$

Consequently, according to the convergence criterion of Weierstrass, from (20) it follows that

$$D(\tilde{x}_n(t), \tilde{x}_{n-1}(t)) \to 0,$$

uniformly on $|t-t_0| \leq \delta$, as $n \to \infty$. Hence, there exists $\tilde{x} : \{t : |t-t_0| \leq \delta \leq a\} \to \mathcal{F}^1$ such that $D(\tilde{x}_n(t), \tilde{x}(t)) \to 0$ uniformly on $|t-t_0| \leq \delta$, as $n \to \infty$. As $D(f(t,\tilde{x}_n(t)), f(t,\tilde{x}(t))) \leq KD(\tilde{x}_n(t), \tilde{x}(t)) \to 0$ uniformly on $|t-t_0| \leq \delta$, as $n \to \infty$, we have

$$\tilde{x}(t) = x_0 \ominus (-1)\int_{t_0}^{t} f(s,\tilde{x}(s))ds.$$

Finally we will prove the uniqueness. In [11], by using the α-differentiability in the first form, was proved that there exists a unique application $x(t)$ defined on $|t-t_0| \leq \delta$ which verifies (14). Now we consider the α-differentiability in the second form and we suppose that there exists $\tilde{y}(t)$, defined on $|t-t_0| \leq \delta$, verifying (15). We need to show that $D(\tilde{x}(t), \tilde{y}(t)) \equiv 0$ for all t such that $|t-t_0| \leq \delta$.
For all $n \in \mathbb{N}$ and $\tilde{x}_n$ defined as in (19), for all $\alpha \in [0,1]$ we have

$$\begin{aligned} & d([\tilde{y}(t)]^\alpha, [\tilde{x}_n(t)]^\alpha) \\ & \leq d\left(\left[\int_{t_0}^t f(s,\tilde{y}(s))ds\right]^\alpha, \left[\int_{t_0}^t f(s,\tilde{x}_{n-1}(s))ds\right]^\alpha\right) \\ & \qquad \leq \int_{t_0}^t K d([\tilde{y}(s)]^\alpha, [\tilde{x}_{n-1}(s)]^\alpha)ds. \end{aligned}$$

Consequently, for all $n \in \mathbb{N}$, we get

$$D([\tilde{y}(t)]^\alpha, [\tilde{x}_n(t)]^\alpha) \leq K \int_{t_0}^t D(\tilde{y}(s), \tilde{x}_{n-1}(s))ds.$$

Note that $D(\tilde{y}(t), x_0) \leq b$ on $|t-t_0| \leq \delta$. Therefore

$$D(\tilde{y}(t), \tilde{x}_1(t)) \leq bK|t-t_0|,$$

on $|t-t_0| \leq \delta$. By an induction procedures, if we assume that

$$D(\tilde{y}(t), \tilde{x}_n(t)) \leq bK^n \frac{|t-t_0|^n}{n!},$$

on $|t-t_0| \leq \delta$, we obtain

$$D(\tilde{y}(t), \tilde{x}_{n+1}(t)) \leq bK^{n+1} \frac{|t-t_0|^{n+1}}{(n+1)!}.$$

Hence, $D(\tilde{y}(t), \tilde{x}_n(t)) = D(\tilde{x}(t), \tilde{x}_n(t)) \to 0$, when $n \to \infty$ and t such that $|t-t_0| \leq \delta$.

Example 4.0.3. Consider the Cauchy problem given by

$$x'(t) = -x(t) + t + 1, \qquad x(0) = C, \qquad (21)$$

where C is a fuzzy interval, that is, an element of $\mathcal{F}^1$ such that each α-level of C is the compact interval $C^\alpha = [c_1, c_2]$. The problem (21) verifies the assumptions of Theorem 4.2 and thus, the existence of solution is guaranteed.

Now we will calculate explicitly the solution. We write the α-levels of x as $x_\alpha(t) = [u_\alpha(t), v_\alpha(t)]$. If $x'(t)$ is the α-derivative of x in the first form, by Theorem 3.1 we have that $x'_\alpha(t) = [u'_\alpha(t), v'_\alpha(t)]$. Then, the fuzzy differential equation in (21) can be expressed as

$$\begin{aligned} [u'_\alpha(t), v'_\alpha(t)] &= -[u_\alpha(t), v_\alpha(t)] + t + 1 \\ &= [-v_\alpha(t) + t + 1, -u_\alpha(t) + t + 1]. \end{aligned}$$

Hence, we obtain the following system of Cauchy problems of ordinary differential equations

$$\begin{aligned} u'_\alpha(t) &= -v_\alpha(t) + t + 1, \qquad u_\alpha(0) = c_1, \\ v'_\alpha(t) &= -u_\alpha(t) + t + 1, \qquad v_\alpha(0) = c_2. \end{aligned}$$

The solutions of the last system are given by

$$u_\alpha(t) = \frac{c_1 + c_2}{2} e^{-t} + \frac{c_1 - c_2}{2} e^t + t$$

and

$$v_\alpha = \frac{c_1 + c_2}{2} e^{-t} + \frac{c_2 - c_1}{2} e^t + t.$$

Thus, the solution of (21) by assuming the α-derivative x' in the first form is

$$x(t) = t + \frac{c_1 + c_2}{2} e^{-t} + \frac{e^t}{2}[C + (-C)], \qquad t \in T.$$

Now, if x' is the α-derivative in the second form, by Theorem 3.1 we have that $x'_\alpha(t) = [v'_\alpha(t), u'_\alpha(t)]$. Then, the fuzzy differential equation in (21) can be expressed as

$$\begin{aligned} [u'_\alpha(t), v'_\alpha(t)] &= -[u_\alpha(t), v_\alpha(t)] + t + 1 \\ &= [-v_\alpha(t) + t + 1, -u_\alpha(t) + t + 1]. \end{aligned}$$

Hence, we obtain the following system of Cauchy problems of ordinary differential equations

$$\begin{aligned} u'_\alpha(t) &= -u_\alpha(t) + t + 1, \qquad u_\alpha(0) = c_1, \\ v'_\alpha(t) &= -v_\alpha(t) + t + 1, \qquad v_\alpha(0) = c_2. \end{aligned}$$

The solutions of the last system are given by

$$u_\alpha(t) = t + c_1 e^{-t} \qquad \text{and} \qquad v_\alpha(t) = t + c_2 e^{-t}.$$

Thus, the another solution of (21) by assuming the α-derivative x' in the second form is

$$\tilde{x}(t) = t + Ce^{-t}, \qquad t \in T.$$

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