ASYMPTOTIC BEHAVIOUR OF THE JACOBI SOBOLEV-TYPE ORTHOGONAL POLYNOMIALS. A NON DIAGONAL CASE

Herbert Dueñas[1], Luis Garza[2]

Abstract

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Consider the following Sobolev type inner product

$$\langle p, q\rangle_s = \int_{-1}^{1} p(x)q(x)(1-x)^{\alpha}(1+x)^{\beta}dx + \mathbb{P}(1)^t \mathbb{A}\mathbb{Q}(1), \tag{1}$$

where p and q are polynomials with real coefficients, $\alpha, \beta > -1$, $\mathbb{P}(x) = (p(x), p'(x))^t$, and

$$\mathbb{A} = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$$

is a positive semidefinite matrix, with $M_{0,} M_1 \geq 0$, and $\lambda \in \mathbb{R}$. The family of polynomials orthogonal with respect to (2), $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$, are called Jacobi Sobolev–type orthogonal polynomials. An expression that relates this family of polynomials with $\{P_n^{\alpha,\beta}\}_{n\geqslant 0}$, the usual Jacobi orthogonal polynomials, was obtained in [8]. Here, we obtain the outer relative asymptotic for $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$, as well as the corresponding Mehler–Heine formula.

Key words: Orthogonal polynomials, Sobolev–type inner products, Jacobi polynomials, Asymptotic behaviour, Zeros.

Resumen

Consideremos el siguiente producto interno de tipo Sobolev

1 Universidad Nacional de Colombia, Bogotá, Colombia. Correo electrónico: haduenasr@unal.edu.co

2 Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Colima, México. Correo electrónico: garzaleg@gmail.com

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$$\langle p, q\rangle_s = \int_{-1}^{1} p(x)q(x)(1-x)^{\alpha}(1+x)^{\beta}dx + \mathbb{P}(1)^t\mathbb{A}\mathbb{Q}(1), \tag{2}$$

donde p y q son polinomios con coeficientes reales, $\alpha, \beta > -1$, $\mathbb{P}(x) = (p(x), p'(x))^t$, y

$$\mathbb{A} = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$$

es una matriz positiva semidefinnida, donde $M_0, M_1 \geq 0$, y $\lambda \in \mathbb{R}$. La familia de los polinomios ortogonales con respecto a (2), $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$, se llaman polinomios del tipo Jacobi Sobolev. Una expresión que relaciona esta familia de polinomios con $\{P_n^{\alpha,\beta}\}_{n\geqslant 0}$, los polinomios ortognales de Jacobi usuales, se obtuvo en [8]. Aquí obtenemos la asintótica relativa exterior para $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$, así como también la correspondiente fórmula de Mehler–Heine.

Palabras clave: Polinomios ortogonales, productos internos del tipo Sobolev, polinomios de Jacobi, comportamiento asintótico, ceros.

1. Introduction

Let us consider the inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(x)q(x)d\sigma(x) + \mathbb{P}(c)^t\mathbb{A}\mathbb{Q}(c), \tag{3}$$

where $d\sigma$ is a nontrivial probability measure supported on the real line, $\mathbb{A} \in \mathbb{R}^{(k,k)}$ is a positive semidefinite matrix, p, q are polynomials with real coefficients, and $\mathbb{Q}(c) = \left(q(c), q'(c), \ldots, q^{(k-1)}(c)\right)^t$. The polynomials orthogonal with respect to (3) are called Sobolev–type orthogonal polynomials, and were introduced in [3].

Most of the work about the Sobolev–type polynomials has been done in the case that the orthogonality measure $d\mu$ corresponds to classical polynomials. In [12], the author studied (3) with $\mathbb{A}$ a diagonal matrix, $d\sigma = x^{\alpha}e^{-x}dx, \alpha > -1$ and $c = 0$. He obtained the second order differential equation satisfied by these polynomials.

The asymptotic properties of orthogonal polynomials with respect to (3) when $\mathbb{A}$ a diagonal matrix have been studied in the literature by many authors. In particular, in [16] the authors consider the outer relative asymptotics, in a more general framework. There, they introduce a Sobolev inner product in terms of a measure of the Nevai class $M(0, 1)$ (a family of measures with a positive a.e absolutely continuous component and supported in $[-1, 1] \cup E$ where E is a discret set with accumulation points in $\{-1, 1\}$ that generalizes the Jacobi weight function) and analyze the outer relative asymptotics of the corresponding orthogonal polynomials according to the location of the point c with respect to the support of the measure. An extension of these results was done for the non diagonal case in [3], as well as in a more general context in [14] for many pass points located outside the support of the measure.

The asymptotic behaviour of the corresponding orthogonal polynomials in the particular case when $k = 2$ was analyzed in [5] and [15], and some properties about the location of the zeros were studied in [13]. On the other hand, for $k \geqslant 2$, $d\sigma = x^{\alpha}e^{-x}dx$, $c = 0$ and $M_0 = M_1 = \cdots = M_{k-2} = 0$, $M_{k-1} > 0$, the same problems were studied in [17], in the framework of the zero distribution, and the study of the asymptotic behaviour was developed in [9].

The nondiagonal case was considered for the first time in [4]. In this paper, the authors considered the Hermite Sobolev–type polynomials, orthogonal with respect to (3) with $\mathbb{A}$ a 2×2 symetric matrix, $d\sigma = e^{-x^2}dx$ and $c = 0$, studying the asymptotic behaviour of such polynomials.

The Laguerre Sobolev–type polynomials, for $k = 2$, $\mathbb{A}$ a nondiagonal matrix, and $d\sigma = x^{\alpha}e^{-x}, \alpha > -1$, were studied in [10], where the authors studied the outer relative asymptotics with respect to the standard Laguerre polynomials, as well as an analog of the Mehler-Heine formula for the rescaled polynomials, and in [11], where the second order differential equation that these polynomials satisfy was deduced.

Finally, the study of the Jacobi Sobolev–type polynomials, orthogonal with respect to (3) where $\mathbb{A}$ is a 2×2 symetric and nondiagonal matrix, $d\sigma = (1-x)^{\alpha}(1+x)^{\beta}dx$ and $c = 1$, started in [8]. There, the authors obtained a connection formula that relates the Jacobi Sobolev–type polynomials with some family of Jacobi polynomials, as well as the holonomic equation that they satisfy. In this contribution, we extend this analysis in order to find their asymptotic behavior. The structure of the manuscript is as follows. Section 2 is devoted to some preliminary results regarding basic theory of classical orthogonal polynomials. In Section 3, we present some of the results obtained in [8], as well as some new properties regarding the zeros of Jacobi Sobolev–type polynomials.

Finally, in Section 4 we present their outer relative asymptotic behavior, as well as their corresponding Mehler-Heine formula.

2. Preliminaries

Let μ be a linear functional defined on $\mathbb{P}$, the linear space of polynomials with real coefficients. If μ is quasi-definite (i.e., if the leading principal submatrices of the Hankel matrix associated with their moments $\mu_n = \langle \mu, x^n \rangle$, $n \geqslant 0$, are non singular), then there exists a family of monic polynomials $\{P_n\}_{n\geqslant 0}$ such that

(i) $\langle \mu, P_n(x)P_m(x) \rangle = 0, \quad n \neq m.$

(ii) $\left\langle \mu, P_n^2(x) \right\rangle \neq 0, \quad n \geqslant 0,$

where the degree of P_n is n. $\{P_n\}_{n\geqslant 0}$ is said to be the sequence of monic polynomials orthogonal with respect to μ, and satisfies the three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geqslant 0, \tag{4}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$ and the recurrence coefficients $\{\beta_n\}_{n\geqslant 0}$, $\{\gamma_n\}_{n\geqslant 1}$ are real numbers with $\gamma_n \neq 0$ for $n \geqslant 0$.

The $n-th$ reproducing kernel associated with $\{P_n\}_{n\geqslant 0}$ is defined by

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\|P_k\|_\mu^2},$$

where $\|P_k\|_\mu^2 = \left\langle \mu, P_k^2(x) \right\rangle$. $K_n(x,y)$ can also be expressed in terms of P_n and P_{n+1}, as follows

$$K_n(x,y) = \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{\|P_n\|_\mu^2 (x-y)}, \tag{5}$$

for $x \neq y$, known in the literature as Christoffel-Darboux formula.

We denote by $K_n^{(i,j)}(x,y)$ the $i-th, j-th$ derivatives of $K_n(x,y)$ with respect to the variables x, y, i.e.,

$$K_n^{(i,j)}(x,y) = \frac{\partial^{i+j}\left(K_n(x,y)\right)}{\partial x^i \partial y^j}.$$

Then, it can be shown that

Proposition 1. [9] *For every* $n \in \mathbb{N}$,

1.
$$K_{n-1}^{(0,j)}(a,a) = \frac{1}{\|P_{n-1}\|_\mu^2 (j+1)} \left(P_{n-1}(a)P_n^{(j+1)}(a) - P_n(a)P_{n-1}^{(j+1)}(a) \right). \tag{6}$$

2.
$$K_{n-1}^{(j,j)}(a,a) = \frac{(j!)^2}{(2j+1)!\, \|P_{n-1}\|_\mu^2} \times \tag{7}$$
$$\left[\left(P_{n-1}(a)P_n^{(2j+1)}(a) + P'_{n-1}(a)P_n^{(2j)}(a) \binom{2j+1}{1} + \cdots + \binom{2j+1}{j} P_{n-1}^{(j)}(a)P_n^{(j+1)}(a) \right) - \left(P_n(a)P_{n-1}^{(2j+1)}(a) + P'_n(a)P_{n-1}^{(2j)}(a) \binom{2j+1}{1} + \cdots + \binom{2j+1}{j} P_n^{(j)}(a)P_{n-1}^{(j+1)}(a) \right) \right].$$

If the linear functional μ is positive definite (i.e., if the leading principal submatrices of the Hankel matrix have positive determinant), then there exists the following integral representation

$$\langle \mu, p(x) \rangle = \int_E p(x) d\sigma(x), \quad p \in \mathbb{P}, \tag{8}$$

where $\sigma(x)$ is a nontrivial probability measure supported on some subset E of the real line. In such a case, there exists a sequence of orthonormal polynomials $\{p_n\}_{n\geqslant 0}$ associated with μ that satisfies

$$\int_E p_n(x)p_m(x) d\sigma(x) = \delta_{n,m}, \quad n, m \geqslant 0. \tag{9}$$

The families of orthogonal polynomials more widely studied are the so-called classical orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel). They can be characterized by the following properties

(a) $\{P_n\}_{n\geqslant 0}$ is classical if and only if there exist sequences $\{a_n\}_{n\geqslant 0}$,$\{b_n\}_{n\geqslant 0}$ and $\{c_n\}_{n\geqslant 0}$, with $c_n \neq 0$, $n \in \mathbb{N}$, such that

$$\phi(x)P'_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x),$$

with $\deg \varphi \leqslant 2$.

(b) They are solutions of a second order differential equation of the form

$$\phi(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where ϕ and τ are polynomials independent of n, which are specific for each family and whose degrees are at most 2 and 1, respectively.

This work deals with (monic) Jacobi polynomials, denoted by $\{P_n^{\alpha,\beta}\}_{n\geqslant 0}$. They are defined by the orthogonality relation

$$\int_{-1}^{1} P_n^{\alpha,\beta} P_m^{\alpha,\beta} d\sigma(x) = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\, n!}{(2n+\alpha+\beta+1)(\Gamma(2n+\alpha+\beta+1))^2}\delta_{n,m},$$

for $n, m \geqslant 0$, where $d\sigma(x) = (1-x)^{\alpha}(1+\alpha)^{\beta}dx$, $\alpha > -1$ and $\beta > -1$. We will denote the corresponding inner product by $\langle \cdot, \cdot \rangle_{\alpha,\beta}$.

The following Proposition summarizes some well known properties (see [6], [7] and [18]) of Jacobi polynomials.

Proposition 2. *Let $\{P_n^{\alpha,\beta}\}_{n\geqslant 0}$ be the sequence of Jacobi monic polynomials.*

1. *For every $n \in \mathbb{N}$,*
$$xP_n^{\alpha,\beta}(x) = P_{n+1}^{\alpha,\beta}(x) + \beta_n^{\alpha,\beta} P_n^{\alpha,\beta}(x) + \gamma_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(x), \tag{10}$$
with
$$\begin{aligned} \beta_n^{\alpha,\beta} &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \gamma_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned}$$
$P_0^{\alpha,\beta}(x) = 1$, and $P_1^{\alpha,\beta}(x) = x + \dfrac{\alpha-\beta}{\alpha+\beta+2}$.
2. *For every $n \in \mathbb{N}$,*
$$\left(1-x^2\right)\left(P_n^{\alpha,\beta}(x)\right)' = a_n^{\alpha,\beta} P_{n+1}^{\alpha,\beta}(x) + b_n^{\alpha,\beta} P_n^{\alpha,\beta}(x) + c_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(x), \tag{11}$$
where
$$\begin{aligned} a_n^{\alpha,\beta} &= -n, \\ b_n^{\alpha,\beta} &= \frac{2(\alpha-\beta)n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \text{ and} \\ c_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}. \end{aligned}$$
3. *For every $n \in \mathbb{N}$, there exists a sequence of real numbers $\{\lambda_n\}_{n\geqslant 0}$ such that $P_n^{\alpha,\beta}(x)$ satisfies the second order linear differential equation*
$$\phi(x)y'' + \psi_{\alpha,\beta}(x)y' = \lambda_n^{\alpha,\beta} y \tag{12}$$
with $\phi(x) = 1-x^2$, $\psi_{\alpha,\beta}(x) = -(\alpha+\beta+2)x + \beta - \alpha$ and $\lambda_n^{\alpha,\beta} = -n(n+1+\alpha+\beta)$.
4. *For every $n \in \mathbb{N}$,*
$$\left(P_n^{\alpha,\beta}(x)\right)' = nP_{n-1}^{\alpha+1,\beta+1}(x). \tag{13}$$
5. *For every $n \in \mathbb{N}$,*
$$P_n^{\alpha,\beta}(x) = P_n^{\alpha+1,\beta}(x) - \frac{2n(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{\alpha+1,\beta}(x). \tag{14}$$
6. *For every $n \in \mathbb{N}$,*
$$K_n(x,1) = A_n P_n^{\alpha+1,\beta}(x), \quad \text{with} \quad A_n = \frac{P_n^{\alpha,\beta}(1)}{\left\|P_n^{\alpha,\beta}\right\|_{\alpha,\beta}^2}. \tag{15}$$
7. *For every $n \in \mathbb{N}$,*
$$P_n^{\alpha,\beta}(1) = 2^n \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+1)}, \tag{16}$$
$$P_n^{\alpha,\beta}(-1) = (-2)^n \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}. \tag{17}$$
8. *For every $n \in \mathbb{N}$,*
$$\left\|P_n^{\alpha,\beta}(x)\right\|_{\alpha,\beta}^2 = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\, n!}{(2n+\alpha+\beta+1)(\Gamma(2n+\alpha+\beta+1))^2}. \tag{18}$$

9. *For every x outside* $[-1,1]$

$$P_n^{\alpha,\beta}(x) \simeq \frac{2^n n!\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}(x-1)^{-\alpha/2}(x+1)^{-\beta/2}\times \left((x+1)^{1/2}+(x-1)^{1/2}\right)^{\alpha+\beta}(2\pi n)^{-1/2}(x^2-1)^{-1/4}\left(x+(x^2-1)^{1/2}\right)^{n+1/2}. \tag{19}$$

10. *(The Mehler–Heine formula) Let J_α be the usual Bessel function of the first kind and* $\widehat{P}_n^{\alpha,\beta}(x) = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}P_n^{\alpha,\beta}(x)$. *Then*

$$\lim_{n\to\infty} n^{-\alpha}\widehat{P}_n^{\alpha,\beta}\left(1-\frac{z^2}{2n^2}\right) = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) \tag{20}$$

uniformly in every compact subset of the complex z-plane.

3. Jacobi Sobolev–type polynomials

Let $p, q \in \mathbb{P}$. We define the following Sobolev–type inner product

$$\langle p, q\rangle_s = \int_{-1}^{1} p(x)q(x)(1-x)^\alpha(1+x)^\beta dx + \mathbb{P}(1)^t\mathbb{A}\mathbb{Q}(1), \tag{21}$$

where $\alpha, \beta > -1$, $\mathbb{P}(x) = [p(x), p'(x)]^t$, and

$$\mathbb{A} = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$$

is a positive semidefinite matrix, with $M_0, M_1 \geq 0$, and $\lambda \in \mathbb{R}$. Notice that if $M_0 = 0$ or $M_1 = 0$, then $\lambda = 0$.

If we denote by $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$ the family of monic polynomials orthogonal with respect to (21), then it can be shown that

Proposition 3. [8] *For $n \geqslant 1$,*

$$\widetilde{P}_n^{\alpha,\beta}(x) = P_n^{\alpha,\beta}(x) - \left(\mathbb{P}_n^{\alpha,\beta}(1)\right)^t(\mathbb{I}+\mathbb{A}\mathbb{K}_{n-1}(1,1))^{-1}\mathbb{A}\begin{pmatrix} K_{n-1}(x,1) \\ K_{n-1}^{(0,1)}(x,1)\end{pmatrix}, \tag{22}$$

where

$$\mathbb{K}_{n-1}(1,1) = \begin{pmatrix} K_{n-1}(1,1) & K_{n-1}^{(1,0)}(1,1) \\ K_{n-1}^{(0,1)}(1,1) & K_{n-1}^{(1,1)}(1,1)\end{pmatrix}.$$

It is also possible to obtain a connection formula that relates $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$ with $\{P_n^{\alpha+2,\beta}\}_{n\geqslant 0}$. Indeed,

Theorem 4. [8] *For every $n \in \mathbb{N}$,*

$$\widetilde{P}_n^{\alpha,\beta}(x) = P_n^{\alpha+2,\beta}(x) + A_{n,\alpha,\beta}P_{n-1}^{\alpha+2,\beta}(x) + B_{n,\alpha,\beta}P_{n-2}^{\alpha+2,\beta}(x), \tag{23}$$

where

$$\begin{aligned} A_{n,\alpha,\beta} &= -\frac{4n(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} - C_{n,\alpha,\beta}, \\ B_{n,\alpha,\beta} &= \frac{4n(n-1)(n+\beta)(n+\beta-1)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)} - D_{n,\alpha,\beta}, \end{aligned}$$

with

$$\begin{aligned} C_{n,\alpha,\beta} &= \frac{\left(\mathbb{P}_n^{\alpha,\beta}(1)\right)^t(\mathbb{I}+\mathbb{A}\mathbb{K}_{n-1}(1,1))^{-1}\Gamma(2n+\alpha+\beta)\mathbb{A}}{2^{n+\alpha+\beta}\Gamma(\alpha+1)\Gamma(n+\beta)(n-2)!}\begin{pmatrix} \frac{1}{n-1} \\ \frac{n+\alpha+\beta}{2(\alpha+1)}\end{pmatrix}, \\ D_{n,\alpha,\beta} &= \frac{\left(\mathbb{P}_n^{\alpha,\beta}(1)\right)^t(\mathbb{I}+\mathbb{A}\mathbb{K}_{n-1}(1,1))^{-1}\Gamma(2n+\alpha+\beta-1)\mathbb{A}}{2^{n+\alpha+\beta-1}\Gamma(\alpha+1)\Gamma(n+\beta-1)(n-2)!(2n+\alpha+\beta)}\begin{pmatrix} 1 \\ -\frac{n(n+\alpha+\beta+1)}{2(\alpha+1)}\end{pmatrix}. \end{aligned}$$

4. Asymptotics of the Jacobi Sobolev–type orthogonal polynomials

In this section, we study the asymptotic behaviour of the Jacobi Sobolev–type orthogonal polynomials $\widetilde{P}_n^{\alpha,\beta}(x)$. From (19) and (23), the following Proposition is straightforward.

Proposition 5. *For every x outside of $[-1,1]$,*

$$\begin{aligned}\widetilde{P}_n^{\alpha,\beta}(x) \quad \simeq \quad & \left[\frac{1}{4}\left(x+(x^2-1)^{1/2}\right)^2 + \frac{A_{n,\alpha,\beta}}{2}\left(x+(x^2-1)^{1/2}\right) + B_{n,\alpha,\beta}\right]\times \\ & \frac{2^{n-2}(n-2)!\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta-1)}\left((x+1)^{1/2}+(x-1)^{1/2}\right)^{\alpha+\beta+2}\times \\ & (x-1)^{-(\alpha+2)/2}(x+1)^{-\beta/2}(x^2-1)^{-1/4}(2\pi n)^{-1/2}\left(x+(x^2-1)^{1/2}\right)^{n-3/2}.\end{aligned}$$

In order to obtain the outer relative asymptotics, we will start by finding explicit expressions for $K_{n-1}(1,1)$, $K_{n-1}^{(0,1)}(1,1)$ and $K_{n-1}^{(1,1)}(1,1)$.

Using (15), we get

$$K_{n-1}(1,1) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta)(n-1)!}.$$

On the other hand, from (6),

$$\begin{aligned}K_{n-1}^{(0,1)}(1,1) &= \frac{1}{2\left\|P_{n-1}^{\alpha,\beta}\right\|_{\alpha,\beta}^2}\left(P_{n-1}^{\alpha,\beta}(1)\left(P_n^{\alpha,\beta}\right)''(1) - P_n^{\alpha,\beta}(1)\left(P_{n-1}^{\alpha,\beta}\right)''(1)\right) \\ &= \frac{1}{2\left\|P_{n-1}^{\alpha,\beta}\right\|_{\alpha,\beta}^2}\left(P_{n-1}^{\alpha,\beta}(1)n(n-1)P_{n-2}^{\alpha+2,\beta+2}(1)\right. \\ &- \left. P_n^{\alpha,\beta}(1)(n-1)(n-2)P_{n-3}^{\alpha+2,\beta+2}(1)\right) \\ &= \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta+2}\Gamma(n+\beta)(n-2)!\Gamma(\alpha+1)\Gamma(\alpha+3)} \\ &= \frac{(n+\alpha+\beta+1)(n-1)}{2(\alpha+2)}K_{n-1}(1,1),\end{aligned}$$

and, from (7), we get

$$\begin{aligned}K_{n-1}^{(1,1)}(1,1) &= \frac{1}{3!\left\|P_{n-1}^{\alpha,\beta}\right\|_{\alpha,\beta}^2}\times\left[\left(P_{n-1}^{\alpha,\beta}(1)\left(P_n^{\alpha,\beta}\right)^{(3)}(1) + 3\left(P_{n-1}^{\alpha,\beta}\right)'(1)\left(P_n^{\alpha,\beta}\right)^{(2)}\right)-\right. \\ & \left.\left(P_n^{\alpha,\beta}(1)\left(P_{n-1}^{\alpha,\beta}\right)^{(3)}(1) + 3\left(P_n^{\alpha,\beta}\right)'(1)\left(P_{n-1}^{\alpha,\beta}\right)^{(2)}\right)\right] \\ &= \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+2)}{3{,}2^{\alpha+\beta+4}\Gamma(\alpha+1)\Gamma(\alpha+3)(n-2)!\Gamma(n+\beta)(2n+\alpha+\beta)}\times \\ & \left[\frac{(n-2)(n+\alpha+\beta+2)[n(n+\alpha+\beta+3)-(n-3)(n+\alpha+\beta)]}{(\alpha+3)}\right. \\ & \left. + \frac{3n(n+\alpha+\beta)[(n-1)(n+\alpha+\beta+2)-(n-2)(n+\alpha+\beta+1)]}{(\alpha+1)}\right] \\ & \times\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+2)k_0}{2^{\alpha+\beta+3}\Gamma(\alpha+2)\Gamma(\alpha+4)(n-2)!\Gamma(n+\beta)},\end{aligned}$$

where $k_0 = \left(-3\alpha - \beta + 2n\alpha + 2n\beta - \alpha^2 - \alpha\beta + n\alpha^2 + n^2\alpha + 2n^2 + n\alpha\beta - 2\right)$. Thus,

$$K_{n-1}^{(1,1)}(1,1) = \frac{(n+\alpha+\beta+1)(n-1)k_0}{4(\alpha+1)(\alpha+2)(\alpha+3)} K_{n-1}(1,1).$$

Therefore, if we define

$$\begin{aligned} k_1 &= \frac{(n+\alpha+\beta+1)(n-1)}{2(\alpha+2)}, \\ k_2 &= \frac{k_0}{2(\alpha+1)(\alpha+3)}, \end{aligned}$$

then we get

Proposition 6. *For every $n \in \mathbb{N}$,*

$$K_{n-1}(1,1) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta)(n-1)!}, \tag{24}$$

$$K_{n-1}^{(0,1)}(1,1) = k_1 K_{n-1}(1,1), \tag{25}$$

$$K_{n-1}^{(1,1)}(1,1) = k_1 k_2 K_{n-1}(1,1). \tag{26}$$

Notice that, when $n \to \infty$,

$$k_1 \simeq \frac{n^2}{2(\alpha+2)}, \tag{27}$$

$$k_2 \simeq \frac{(\alpha+2)n^2}{2(\alpha+1)(\alpha+3)}. \tag{28}$$

On the other hand,

$$\begin{aligned} \mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix} \begin{pmatrix} K_{n-1}(1,1) & k_1 K_{n-1}(1,1) \\ k_1 K_{n-1}(1,1) & k_1 k_2 K_{n-1}(1,1) \end{pmatrix}, \\ &= K_{n-1}(1,1) \begin{pmatrix} \frac{1}{K_{n-1}(1,1)} & 0 \\ 0 & \frac{1}{K_{n-1}(1,1)} \end{pmatrix} + K_{n-1}(1,1) \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix} \begin{pmatrix} 1 & k_1 \\ k_1 & k_1 k_2 \end{pmatrix}, \\ &= K_{n-1}(1,1) \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} Y_1 &= \tfrac{1}{K_{n-1}(1,1)} + M_0 + \lambda k_1, \\ Y_2 &= (M_0 + \lambda k_2) k_1, \\ Y_3 &= \lambda + M_1 k_1, \\ Y_4 &= \tfrac{1}{K_{n-1}(1,1)} + (\lambda + M_1 k_2) k_1, \end{aligned} \tag{29}$$

and thus

$$(\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1))^{-1} = \frac{K_{n-1}(1,1)}{|\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)|} \begin{pmatrix} Y_4 & -Y_2 \\ -Y_3 & Y_1 \end{pmatrix},$$

Also,

$$\begin{aligned} |\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)| &= (K_{n-1}(1,1))^2 \lambda^2 k_1^2 - (K_{n-1}(1,1))^2 k_2 k_1 \lambda^2 - (K_{n-1}(1,1))^2 M_0 M_1 k_1^2 \\ &+ (K_{n-1}(1,1))^2 M_0 M_1 k_2 k_1 + 2K_{n-1}(1,1)\lambda k_1 + M_1 k_2 K_{n-1}(1,1) k_1 \\ &+ M_0 K_{n-1}(1,1) + 1. \end{aligned}$$

We have two possible situations. First, if $|\mathbb{A}| \neq 0$,

$$\begin{aligned} |\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)| &\simeq (K_{n-1}(1,1))^2 \lambda^2 k_1^2 - (K_{n-1}(1,1))^2 k_2 k_1 \lambda^2 \\ &- (K_{n-1}(1,1))^2 M_0 M_1 k_1^2 + (K_{n-1}(1,1))^2 M_0 M_1 k_2 k_1 \\ &= (K_{n-1}(1,1))^2 \left(|\mathbb{A}| k_1 k_2 - |\mathbb{A}| k_1^2\right) \\ &\simeq |\mathbb{A}| \frac{n^2}{2(\alpha+2)} \frac{n^2}{(\alpha+1)(\alpha+2)(\alpha+3)} (K_{n-1}(1,1))^2 . \end{aligned}$$

On the other hand, if $|\mathbb{A}| = 0$, $M_1 \neq 0$,

$$\begin{aligned} |\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)| &= 2K_{n-1}(1,1)\lambda k_1 + M_1 k_1 k_2 K_{n-1}(1,1) \\ &+ M_0 K_{n-1}(1,1) + 1, \end{aligned}$$

and, as a consequence,

$$|\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)| \simeq M_1 k_1 k_2 K_{n-1}(1,1).$$

Thus, we get

Proposition 7.

(*i*) *If* $|\mathbb{A}| \neq 0$,

$$|\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)| \simeq \frac{n^4 |\mathbb{A}|}{2(\alpha+1)(\alpha+2)^2(\alpha+3)} (K_{n-1}(1,1))^2 . \tag{30}$$

(*ii*) *If* $|\mathbb{A}| = 0$, $M_1 \neq 0$,

$$|\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)| \simeq \frac{M_1 n^4 K_{n-1}(1,1)}{4(\alpha+1)(\alpha+3)}. \tag{31}$$

Now, using the expressions obtained above, we proceed to estimate the asymptotics for the coefficients of the connection formula, $A_{n,\alpha,\beta}$ and $B_{n,\alpha,\beta}$. We have

$$\begin{aligned} &\begin{pmatrix} P_n^{\alpha,\beta}(1) & \left(P_n^{\alpha,\beta}\right)'(1) \end{pmatrix} \begin{pmatrix} Y_4 & -Y_2 \\ -Y_3 & Y_1 \end{pmatrix} \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix} \begin{pmatrix} \frac{1}{n-1} \\ \frac{n+\alpha+\beta}{2(\alpha+1)} \end{pmatrix} \\ = \; &\frac{1}{n-1} \left[\left(P_n^{\alpha,\beta}(1) Y_4 - \left(P_n^{\alpha,\beta}\right)'(1) Y_3 \right) M_0 + \left(-P_n^{\alpha,\beta}(1) Y_2 + \left(P_n^{\alpha,\beta}\right)'(1) Y_1 \right) \lambda \right] + \\ &\left[\left(P_n^{\alpha,\beta}(1) Y_4 - \left(P_n^{\alpha,\beta}\right)'(1) Y_3 \right) \lambda + \left(-P_n^{\alpha,\beta}(1) Y_2 + \left(P_n^{\alpha,\beta}\right)'(1) Y_1 \right) M_1 \right] \frac{n+\alpha+\beta}{2(\alpha+1)}. \end{aligned}$$

Thus, using (16)

$$\begin{aligned} C_{n,\alpha,\beta} = \; &\frac{\Gamma(2n+\alpha+\beta)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)^2\Gamma(n+\beta)(n-2)!\, |\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)|\, \Gamma(2n+\alpha+\beta+1)} \times \\ &\left[\begin{array}{c} \frac{1}{n-1} \left(2 \left(\frac{M_0}{K_{n-1}(1,1)} + k_2 k_1 |\mathbb{A}| \right) + \frac{n(n+\alpha+\beta+1)}{(\alpha+1)} \left(\frac{\lambda}{K_{n-1}(1,1)} - |\mathbb{A}| k_1 \right) \right) \\ + \frac{n+\alpha+\beta}{2(\alpha+1)} \left(2 \left(\frac{\lambda}{K_{n-1}(1,1)} - |\mathbb{A}| k_1 \right) + \frac{n(n+\alpha+\beta)}{(\alpha+1)} \left(\frac{M_1}{K_{n-1}(1,1)} + |\mathbb{A}| \right) \right) \end{array} \right], \end{aligned}$$

and using (24) and (29), for $|\mathbb{A}| \neq 0$,

$$C_{n,\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)\, |\mathbb{A}|\, k_1 \left(-n^2 + 2(\alpha+1)k_2 \right)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta)(n-1)!\, |\mathbb{I} + \mathbb{A}\mathbb{K}_{n-1}(1,1)|\, (2n+\alpha+\beta)}.$$

From (27) and (30) we obtain

$$C_{n,\alpha,\beta} \simeq -\frac{2^{\alpha+\beta+1}\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(n+\beta)(n-1)!}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta)},$$

and, using the fact that

$$\Gamma(x) \simeq \sqrt{2\pi} x^{x-1/2} e^{-x},$$

we get

$$C_{n,\alpha,\beta} \simeq -e^{2(\alpha+1)}2^{\alpha+\beta+1}\Gamma(\alpha+2)\Gamma(\alpha+3)\left(\frac{(n+\beta)n}{(n+\alpha+1)(n+\alpha+\beta+1)}\right)^n \times \frac{(n+\beta)^{\beta-1/2}n^{-1/2}}{(n+\alpha+1)^{\alpha+1/2}(2n+\alpha+\beta)(n+\alpha+\beta+1)^{\alpha+\beta+1/2}}.$$

Given that

$$\lim_{n\to\infty}\left(\frac{(n+\beta)n}{(n+\alpha+1)(n+\alpha+\beta+1)}\right)^n = e^{-2\alpha-2},$$

then,

$$C_{n,\alpha,\beta} \simeq \frac{L_1}{n^{3+2\alpha}}, \qquad (32)$$

where

$$L_1 = -2^{\alpha+\beta}\Gamma(\alpha+2)\Gamma(\alpha+3).$$

In a similar way, we can show that there exists a constant L_2 such that

$$D_{n,\alpha,\beta} \simeq \frac{L_2}{n^{3+2\alpha}}, \qquad (33)$$

with

$$L_2 = -2^{\alpha+\beta}(\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha+3).$$

For $|\mathbb{A}| = 0$, $M_1 \neq 0$, on the other hand, using an analogous process, we can find constants

$$T_1 = 2^{\alpha+\beta+1}(\alpha+3)\Gamma(\alpha+1)\Gamma(\alpha+2),$$
$$T_2 = -2^{\alpha+\beta+1}(\alpha+3)\left(\Gamma(\alpha+2)\right)^2,$$

such that

$$C_{n,\alpha,\beta} \simeq \frac{T_1}{n^{3+2\alpha}},$$

$$D_{n,\alpha,\beta} \simeq \frac{T_2}{n^{3+2\alpha}}.$$

In other words, the value of $|\mathbb{A}|$ has no effect on the behavior of the coefficients of the connection formula. As a consequence

Proposition 8. *We have*

(*i*) $\lim_{n\to\infty} A_{n,\alpha,\beta} = -1$,

(*ii*) $\lim_{n\to\infty} B_{n,\alpha,\beta} = \frac{1}{4}$.

Now, we are ready to estimate the ratio asymptotic for $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$. Indeed,

Theorem 9. *Let $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$ be the sequence of monic polynomials orthogonal with respect to (21). Then,*

$$\lim_{n\to\infty}\frac{\widetilde{P}_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(x)} = 1,$$

uniformly on every compact subset of $\mathbb{C}\setminus[-1,1]$.

Proof. Using (19) and (23) we get

$$\begin{aligned}\frac{\widetilde{P}_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(x)} &\simeq \frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)(x-1)^{-1}\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\ &+ A_{n,\alpha,\beta}\frac{(n+\alpha+\beta+1)\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2(2\pi(n-1))^{-1/2}}{2n(x-1)\left(x+(x^2-1)^{1/2}\right)(2\pi n)^{-1/2}} \\ &+ B_{n,\alpha,\beta}\frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2(2\pi n)^{1/2}}{4n(n-1)(x-1)\left(x+(x^2-1)^{1/2}\right)^2(2\pi(n-2))^{1/2}},\end{aligned}$$

where x is outside $[-1,1]$. Therefore, as $n\to\infty$,

$$\begin{aligned}\lim_{n\to\infty}\frac{\widetilde{P}_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(x)} &= \frac{\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2}{4(x-1)} - \frac{\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2}{2(x-1)\left(x+(x^2-1)^{1/2}\right)} \\ &+ \frac{\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2}{4(x-1)\left(x+(x^2-1)^{1/2}\right)^2} \\ &= \frac{\left((x+1)^{1/2}+(x-1)^{1/2}\right)^2}{4(x-1)}\left(1-\frac{1}{x+(x^2-1)^{1/2}}\right)^2 \\ &= \frac{\left(x+(x^2-1)^{1/2}-1\right)^2}{2(x-1)\left(x+(x^2-1)^{1/2}\right)} = 1.\end{aligned}$$

In order to find a corresponding Mehler–Heine formula for the Jacobi Sobolev–type orthogonal polynomials $\{\widetilde{P}_n^{\alpha,\beta}\}_{n\geqslant 0}$, we use the following notation

$$Q_n^{\alpha,\beta}(x) = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}\widetilde{P}_n^{\alpha,\beta}(x).$$

From (23) we get

$$\widetilde{P}_n^{\alpha,\beta}(x) = P_n^{\alpha,\beta}(x) + C_{n,\alpha,\beta}P_{n-1}^{\alpha+2,\beta}(x) + D_{n,\alpha,\beta}P_{n-2}^{\alpha+2,\beta}(x), \tag{34}$$

Thus, from (34)

$$\begin{aligned} Q_n^{\alpha,\beta}(x) &= \widehat{P}_n^{\alpha,\beta}(x) + \frac{n+\alpha+\beta+1}{2n}C_{n,\alpha,\beta}\widehat{P}_{n-1}^{\alpha+2,\beta}(x) \\ &+ \frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}{4n(n-1)}D_{n,\alpha,\beta}\widehat{P}_{n-2}^{\alpha+2,\beta}(x), \end{aligned}$$

and therefore, using (32) and (33) we get

$$\begin{aligned} n^{-\alpha}Q_n^{\alpha,\beta}\left(1-\frac{z^2}{2n^2}\right) &\simeq n^{-\alpha}\widehat{P}_n^{\alpha,\beta}\left(1-\frac{z^2}{2n^2}\right) + n^{-\alpha}\frac{L_1}{n^{3+2\alpha}}\frac{n+\alpha+\beta+1}{2n}\widehat{P}_{n-1}^{\alpha+2,\beta}\left(1-\frac{z^2}{2n^2}\right) \\ &+n^{-\alpha}\frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}{4n(n-1)}\frac{L_2}{n^{3+2\alpha}}\widehat{P}_{n-2}^{\alpha+2,\beta}\left(1-\frac{z^2}{2n^2}\right) \\ &= n^{-\alpha}\widehat{P}_n^{\alpha,\beta}\left(1-\frac{z^2}{2n^2}\right) + \frac{L_1}{n^{1+2\alpha}}\frac{n+\alpha+\beta+1}{2n}n^{-(\alpha+2)}\widehat{P}_{n-1}^{\alpha+2,\beta}\left(1-\frac{z^2}{2n^2}\right) \\ &+\frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}{4n(n-1)}\frac{L_2}{n^{1+2\alpha}}n^{-(\alpha+2)}\widehat{P}_{n-2}^{\alpha+2,\beta}\left(1-\frac{z^2}{2n^2}\right). \end{aligned}$$

As a consequence, we get

Theorem 10. *Let $\{Q_n^{\alpha,\beta}\}_{n\geqslant 0}$ be the sequence of Jacobi Sobolev–type orthogonal polynomials defined previously. Then*

1. *If $\alpha = -1/2$*
$$\lim_{n\to\infty} n^{-\alpha}Q_n^{\alpha,\beta}\left(1-\frac{z^2}{2n^2}\right) = (z/2)^{-\alpha}J_\alpha(z) + \left(\frac{L_1}{2}+L_2\right)J_{\alpha+2}(z).$$
2. *If $\alpha > -1/2$*
$$\lim_{n\to\infty} n^{-\alpha}Q_n^{\alpha,\beta}\left(1-\frac{z^2}{2n^2}\right) = (z/2)^{-\alpha}J_\alpha(z),$$

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