A UNIQUE CONTINUATION RESULT FOR A GENERALIZED KDV TYPE EQUATION WITH VARIABLE COEFFICIENTS

By

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Abstract

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In this paper we establish a unique continuation result for a generalized KdV type equation with variable coefficients of the form

$$u_t + \gamma\partial_x^3 u + kr_1(x,t)u^{k-1}\partial_x u + r_2(x,t)\partial_x u + r_3(x,t)u = 0, \quad (k \geq 2)$$

in the following sense. If u is a sufficiently smooth solution such that $\operatorname{supp} u(x,t) \subseteq [-B,B] \times [-T,T]$, then u must be necessarily the zero solution, assuming some decay in the Fourier transform of the coefficients $r_i(x,t)$ with respect to the spatial variable. The result follows by adapting and extending the techniques developed by **J. Bourgain** in [1], used to obtain a unique continuation result for a generalized KdV type equation (constant coefficient case).

Key words: KdV type equation with variable coefficients, unique continuation.

Resumen

En este trabajo establecemos un resultado de continuación única para la ecuación del tipo KdV generalizada

$$u_t + \gamma\partial_x^3 u + kr_1(x,t)u^{k-1}\partial_x u + r_2(x,t)\partial_x u + r_3(x,t)u = 0, \quad (k \geq 2)$$

en el siguiente sentido. Si u una solución suficientemente suave tal que $\operatorname{supp} u(x,t) \subseteq [-B,B] \times [-T,T]$, entonces u necesariamente debe ser cero, bajo la suposición de que la transformada de Fourier de los coeficientes $r_i(x,t)$ tiene algún tipo de decaimiento con respecto a la variable x. El resultado es obtenido adaptando y extendiendo las técnicas desarrolladas por **J. Bourgain** en [1], utilizadas para obtener el resultado de continuación única para la ecuación generalizada KdV (coeficientes constantes).

Palabras clave: Ecuaciones de tipo KdV con coeficientes variables, continuación única.

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1. Introduction

In a recent work, **Jean Bourgain** in [1] proved that solutions u sufficiently smooth with $\operatorname{supp} u(x,t) \subseteq [-B,B] \times [-T,T]$ of the generalized Korteweg–De Vries type equation

$$u_t + \partial_x^3 u + \partial_x F(u) = 0 \quad \text{en} \quad \mathbb{R}^{1+1}, \qquad \text{(FKdV)}$$

with F being a real polynomial, must be trivial ($u \equiv 0$ in $\mathbb{R} \times \mathbb{R}$).

In order to have an insight into **Bourgain**'s proof, we want to point out that the main idea is to take advantage of two facts: 1.- The Fourier transform with respect to the spatial variable x of a continuous real function $u(t)(x) = u(x,t)$ defined in $\mathbb{R} \times [-T,T]$ having

$$\operatorname{supp} u(t)(\cdot) \subseteq [-B,B], \quad \text{for } t \in [-T,T]$$

can be extended to $\mathbb{C} \times [-T,T]$, having exponential order.

2. The derivative of entire functions having exponential order and being bounded in the real axis can be some how controlled. These facts are evident in the following complex analysis results.

Theorem 1.1. [Paley-Wiener Theorem] *Let $u(t)(x) = u(x,t)$ be a continuous real function defined in $\mathbb{R} \times [-T,T]$ such that*

$$\operatorname{supp} u(t)(\cdot) \subseteq [-B,B], \quad \text{for all } t \in I = [-T,T].$$

Then, the Fourier transform of $u(t)$ with respect to the spatial variable x,

$$\widehat{u(t)}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x} u(t)(x)dx,$$

has a unique analytic extension to $\mathbb{C}$. Moreover, the extension has exponential order. In other words, there is a positive constant $k > 0$ such that for $t \in I$,

$$\left|\widehat{u(t)}(\lambda + i\sigma)\right| \leq k e^{|\sigma|B}, \quad \lambda, \sigma \in \mathbb{R}. \qquad (1)$$

Theorem 1.2. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function such that*

$$|\phi(\lambda + i\sigma)| \leq \kappa e^{|\sigma|B}, \quad \lambda, \sigma \in \mathbb{R}. \qquad (2)$$

Then there is $\mu > 0$ such that for any $\lambda_1 > 0$,

$$|\phi'(\lambda_1)| \leq \mu B\left(\sup_{|\xi|\geq\lambda_1} |\phi(\xi)|\right)\left[1 + \left|\log\left(\sup_{|\xi|\geq\lambda_1} |\phi(\xi)|\right)\right|\right].$$

Theorem 1.3. *Let $\phi : \mathbb{C} \to \mathbb{C}$ as in previous Theorem. Then there is $\mu > 1$ such that if*

$$\sup_{|\xi|\geq\lambda_1} |\phi(\xi + i\sigma)| \leq 2 \sup_{|\xi|\geq\lambda_1} |\phi(\xi)|, \qquad (3)$$

holds for $\lambda_1 > 0$ and $\sigma \in \mathbb{R}$, then

$$\sup_{|\xi|\geq\lambda_1} |\phi'(\xi + i\sigma)| \leq \mu B\left(\sup_{|\xi|\geq\lambda_1} |\phi(\xi)|\right)\left[1 + \left|\log\left(\sup_{|\xi|\geq\lambda_1} |\phi(\xi)|\right)\right|\right]. \qquad (4)$$

Finally, to related these results, we must establish for functions u as in the Paley–Wiener Theorem for $\lambda_1 > 0$, and $t_0 \in I$ fixed, that we have the estimate

$$\sup_{|\xi|\geq\lambda_1} |\widehat{u(t_0)}(\xi + i\sigma)| \leq 2 \sup_{|\xi|\geq\lambda_1} |\widehat{u(t_0)}(\xi)|, \qquad (5)$$

for $\sigma \in \mathbb{R}$ with $|\sigma|$ small enough (see Lemma (3.2) below).

In order to illustrate the situation, let us suppose that u is a smooth solution of the linear equation

$$u_t + \gamma u_{xxx} = 0,$$

with $\operatorname{supp} u(t) \subset [-B,B] \times I$, then by the Paley–Wiener Theorem we conclude that $\widehat{u(t)}$ has an analytic extension in $\mathbb{C}$ and there exists $\kappa > 0$ such that for all $t \in I$, the function $\widehat{u(t)}(\lambda + i\sigma)$ has the exponential order (1). Now, using the semigroup associated with the linear equation, we know for $t_1 \in I$ that the solution $u(t)(\cdot)$ can be expressed in terms of the Fourier transform as

$$\widehat{u(t)}(\lambda) = e^{i\gamma\lambda^3\Delta t}\widehat{u(t_1)}(\lambda), \quad \Delta t = t - t_1.$$

Moreover, we have that the extension has the form

$$\widehat{u(t)}(\lambda + i\sigma) = e^{i\gamma(\lambda+i\sigma)^3\Delta t}\widehat{u(t_1)}(\lambda + i\sigma).$$

Now, applying the triangular inequality and the generalized Mean Value Theorem, we conclude that there is some $0 \leq |\sigma_0| \leq |\sigma|$ such that

$$\begin{aligned}|\widehat{u(t)}(\lambda + i\sigma)| &\geq e^{\gamma(\sigma^3 - 3\sigma\lambda^2)\Delta t}\Big[|\widehat{u(t_1)}(\lambda)| \\ &\qquad - |\widehat{u(t_1)}(\lambda + i\sigma) - \widehat{u(t_1)}(\lambda)|\Big] \\ &\geq e^{\gamma(\sigma^2 - 3\lambda^2)\sigma\Delta t}\left[\left|\widehat{u(t_1)}(\lambda)\right| - |\sigma|\left|\left(\widehat{u(t_1)}\right)'(\lambda + i\sigma_0)\right|\right].\end{aligned}$$

Since we know that the derivative of an entire function with exponential decay and bounded in the real axis (see Theorem (1.3)) is controlled, then for $|\sigma|$ small enough, we have that

$$Ce^{|\sigma|B} \geq |\widehat{u(t)}(\lambda + i\sigma)| \geq \frac{1}{2}e^{\gamma(\sigma^2 - 3\lambda^2)\sigma\Delta t}|\widehat{u(t_1)}(\lambda)|,$$

which implies that

$$e^{|\sigma|B-\gamma\sigma^3\Delta t+3\gamma\lambda^2\sigma\Delta t} \geq \frac{1}{2C}|\widehat{u(t_1)}(\lambda)|.$$

As a consequence of this, if for some t_1 we have that $|\widehat{u(t_1)}(\lambda)| > 0$ for λ sufficiently large and we are able to choose $\gamma\sigma\Delta t < 0$, then we will reach a contradiction since the left hand side is converging rapidly to zero, as λ tends to ∞. Essentially, this is also the situation while trying to obtain a unique continuation result for the equation

$$u_t + \gamma\partial_x^3 u = G(t)(x),$$

in case $G(t)(x) = -\partial_x F(u(x))$ where F is a polynomial (corresponding to the (FKdV) equation), and in case

$$\begin{aligned} G(t)(x) &= -(kr_1(t)(x)u^{k-1}\partial_x u + r_2(t)(x)\partial_x u + r_3(t)(x)u) \\ &= -\left(r_1(t)(x)\partial_x(u^k) + r_2(t)(x)\partial_x u + r_3(t)(x)u\right), \end{aligned}$$

which corresponds to the equation we are considering in this work.

The equation (FKdV) is well known in the literature as the generalized Korteweg–De Vries equation when $F(s) = s^k$ with $k \geq 2$. In this case, the equation takes the form

$$u_t + \gamma\partial_x^3 u + ku^{k-1}\partial_x u = 0 \text{ en } \mathbb{R}^{1+1}. \qquad \text{(gKdV)}$$

In particular, if we assume that u and v are sufficiently smooth solutions of the equation (gKdV), then $w = u-v$ is a solutions of the generalized KdV type equation with variable coefficients

$$w_t + \gamma\partial_x^3 w + \alpha_1(x,t)\partial_x w + \alpha_2(x,t)w = 0,$$

where α_i depends on u and v. Then if we have that $u = v$ in $\mathbb{R}^2 \setminus [-B,B] \times I$, we obtain that

$$\operatorname{supp} w(x,t) \subseteq [-B,B] \times I.$$

Thus if there is a unique continuation result for generalized KdV type equations with variable coefficients analogous to the **Bourgain**'s result, we conclude that $w = 0$ in $\mathbb{R} \times I$. In other words, $u \equiv v$ in $\mathbb{R} \times I$.

We want to point out that the unique continuation problem for KdV type equation has brought the attention to well known mathematicians. In fact, **J. C. Saut** and **B. Scheurer** in ([6]), using estimates of Carleman, proved that if u satisfies the linear equation

$$u_t + u_{xxx} + r_2(x,t)u_{xx} + r_1(x,t)u_x + r_0(x,t)u = 0$$

en $(a,b) \times (t_1,t_2)$, and u is zero in an open set $\Omega \subset (a,b) \times (t_1,t_2)$, then u is zero in the horizontal component of Ω given by

$$\left\{(x,t) \in (a,b) \times (t_1,t_2) : (y,t) \in \Omega \text{ for some } y \in (a,b)\right\}.$$

Moreover, if u is a sufficiently smooth solution of the generalized KdV equation (gKdV) with $\operatorname{supp} u \subset \mathbb{R} \setminus (a,b)$ for all $t \in (t_1,t_2)$, then $u \equiv 0$ in $(a,b) \times (t_1,t_2)$.

On the other hand, **B. Zhang** in [7] showed that the unique smooth solution u of the Korteweg–de Vires equation

$$u_t + u_{xxx} + 2uu_x = 0 \text{ en } \mathbb{R} \times \mathbb{R} \qquad \text{(KdV)}$$

is $u \equiv 0$ in $\mathbb{R} \times \mathbb{R}$, if there are times $t_1 < t_2$ such that for some $\alpha \in \mathbb{R}$,

$$\operatorname{supp} u(t_j) \subset (-\infty, \alpha), j = 1, 2,$$

or

$$\operatorname{supp} u(t_j) \subset (\alpha, \infty), j = 1, 2.$$

In particular, if $u(x,t)$ is a smooth solution of the equation (KdV) vanishing in the open set of $\mathbb{R} \times \mathbb{R}$, then u must be the zero function in $\mathbb{R} \times \mathbb{R}$. Through the Miura transformation, **B. Zhang** obtained a similar result for the modified (KdV) equation

$$v_t + v_{xxx} - 6v^2 v_x = 0 \text{ en } \mathbb{R} \times \mathbb{R}.$$

Zhang's approach is based on the inverse scattering transform theory and properties of the Hardy spaces H_+^2.

C. Kenig, G. Ponce and **L. Vega** in [4] combining decay properties of solutions and **J. C. Saut** and **B. Scheurer** results proved that sufficiently smooth solutions u of the generalized (gKdV) equation are zero, whenever

$$\operatorname{supp} u(t_j) \subset (-\infty, \alpha) \text{ or } \operatorname{supp} u(t_j) \subset (\alpha, \infty) \quad (j = 1, 2).$$

In this paper, we are interested in obtaining a unique continuation result for a generalized Korteweg–de Vries equation with variable coefficients

$$u_t + \gamma\partial_x^3 u + kr_1(x,t)u^{k-1}\partial_x u + r_2(x,t)\partial_x u + r_3(x,t)u = 0, \quad (6)$$

for ($k \geq 2$). This model is appropriated to describe large–amplitude internal waves in a variable medium, as is the case of the coastal waters of the ocean. (see [2] and [3]). It is important to point out that the existence of sufficiently smooth solutions for the generalized Korteweg–de Vries equation with variable coefficients (6) follows by the remark (c) to Theorem 1.3 in **Kenig** et. al. paper [4] (see also [5]). In this case, we write the equation (6) as

$$u_t + \gamma\partial_x^3 u + G(x,t,u,\partial_x u) = 0,$$

where $G(x,t,u,v) = kr_1(x,t)u^{k-1}v + r_2(x,t)v + r_3(x,t)u$.

The result is obtained by adapting and extending the techniques developed by **J. Bourgain** in [1]. It is important to mention that the variable coefficients case is rather different from the constant coefficient case since the nonlinear estimates require a more careful analysis. We will obtain a unique continuation result by imposing some restriction to the variables coefficients $r_i(t)(\cdot)$, which are related with the need of having the global bound

$$\left|\widehat{r_i(t)}(\lambda)\right| \leq \frac{ce^{-|\lambda|}}{1+\lambda^2}. \tag{7}$$

We note that functions $r_i(t)(\cdot)$ satisfying the estimate (7) are easily obtained. For instance, if $r_i(t)(\cdot)$ is sufficiently smooth, then it is easy to see that the bound (7) holds locally, and we also know that the Fourier transform $\widehat{r_i(t)}(\lambda)$ decays rapidly to zero, as $\lambda \to \infty$.

This paper is organized as follows. In section 2, we extend **Bourgain**'s results to the variable coefficients case. We exhibit a class of variable coefficients $r_i(t)(\cdot)$ having the global bound (7) in the spatial variable. In section 3, we prove the unique continuation result for equation (6).

2. Extension of Bourgain's Results

In this section we will establish the extension of **Bourgain**'s results to study the case of variable coefficients, including nontrivial examples of coefficients $r_i(t)(\cdot)$ satisfying the global bound (7). In particular, we obtain a variation of the Lemma in page 440 of **J. Bourgain**'s work in [1] for the variable coefficients case. This result will be clever in the next section to get the extension of the unique continuation result in the case of variable coefficients.

Hereafter we will assume the same type of hypotheses as in **J. Bourgain**'s paper [1]. We say that a function u defined in $\mathbb{R}\times I$ is sufficiently smooth, if the partial derivatives u_t and $\partial_x^4 u$ exist and are continuous. It is clear that this is not a restriction at all since we are dealing with solutions of partial differential equations with smooth coefficients, for which the smoothness is guaranteed at least locally in time.

Definition 2.1. *Let u be a sufficiently smooth function in $\mathbb{R}\times I$ such that for any $t\in I$,*

$$\int_{\mathbb{R}} \left(|u(t)(x)| + |\partial_x^4 u(t)(x)|\right)dx \leq M. \tag{8}$$

Let u^* and a_u be the functions defined as

$$u^*(\lambda) = \sup_{t\in I} |\widehat{u(t)}(\lambda)|, \quad \lambda\in\mathbb{R}, \tag{9}$$

$$a_u(\lambda) = \sup_{|\xi|\geq|\lambda|} u^*(\xi), \quad \lambda\in\mathbb{R}. \tag{10}$$

For the sake of completeness, we include and/or complete the proof of some result of **J. Bourgain** in ([1]), mainly those results that we must extend. Hereafter, $a_u \equiv a$, unless we want to emphasize the function u.

Lemma 2.1. *Let u be a sufficiently smooth function in $\mathbb{R}\times I$ satisfying (8), Then the function a is an even, bounded, nonnegative, and decreasing in the following sense:*

$$a(\lambda_2) \leq a(\lambda_1), \quad \text{if} \quad |\lambda_1| \leq |\lambda_2|. \tag{11}$$

Moreover, we have that

$$\lim_{|\lambda|\to\infty} a(\lambda) = 0. \tag{12}$$

Proof. We claim that u^* and a are well defined. From the hypotheses on u, there exists $M>0$ such that for any $t\in I$ and $\lambda\in\mathbb{R}$,

$$|\widehat{u(t)}(\lambda)| \leq \int_{\mathbb{R}} |u(t)(x)|dx \leq M. \tag{13}$$

Thus we conclude that u^* is well defined. On the other hand,

$$i^4\lambda^4\widehat{u(t)}(\lambda) = \widehat{\partial_x^4 u(t)}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x}\partial_x^4 u(t)(x)dx,$$

Thus we obtain

$$\lambda^4|\widehat{u(t)}(\lambda)| \leq \int_{\mathbb{R}} |\partial_x^4 u(t)(x)|dx \leq M. \tag{14}$$

Using (13) and (14), we have for $t\in I$ that

$$(1+\lambda^4)|\widehat{u(t)}(\lambda)| \leq 2M,$$

implying that for some $C>0$ and for any $\lambda\in\mathbb{R}$,

$$u^*(\lambda) < \frac{C}{1+\lambda^4}.$$

Then for $|\xi|\geq|\lambda|$, we have that

$$u^*(\xi) < \frac{C}{1+|\xi|^4} \leq \frac{C}{1+\lambda^4}.$$

In other words, we have for $\lambda\in\mathbb{R}$ that

$$a(\lambda) \leq \frac{C}{1+\lambda^4}.$$

Moreover, for some $C_1>0$,

$$a(\lambda) < \frac{C_1}{1+\lambda^4} < C_1, \tag{15}$$

implying that the function a is integrable in $\mathbb{R}$ and bounded. Note that a is an even function, since

$$a(-\lambda) = \sup_{|\xi|\geq|-\lambda|} u^*(\xi) = \sup_{|\xi|\geq|\lambda|} u^*(\xi) = a(\lambda).$$

Now, for $|\lambda_1| \leq |\lambda_2|$, we have that

$$a(\lambda_2) = \sup_{|\xi|\geq|\lambda_2|} u^*(\xi) \leq \sup_{|\xi|\geq|\lambda_1|} u^*(\xi) = a(\lambda_1),$$

proving (11). Finally, from (15) we obtain that

$$\lim_{|\lambda|\to\infty} a(\lambda) = 0. \qquad \square$$

Lemma 2.2. *Let u be a sufficiently smooth function in $\mathbb{R}\times I$ satisfying (8), then there exists $B_1 > 0$ such that for $\lambda_0 > 0$ fixed,*

$$a(\lambda) < \frac{2B_1}{1+\lambda^2}\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}, \tag{16}$$

whenever $|\lambda| \leq \lambda_0$ and for any $Q > 0$.

Proof. First note that for some $B_1 > 0$ (which depends only on the regularity of u), we have for $\lambda \in \mathbb{R}$ that

$$a(\lambda) < \frac{B_1}{1+\lambda^4} \quad \text{and} \quad a(\lambda) < \frac{B_1}{1+\lambda^2}. \tag{17}$$

In fact, from (15) we know that there is $C_1 > 0$ such that

$$a(\lambda) < \frac{C_1}{1+\lambda^4}.$$

Then we also have for $|\lambda| > 1$ that

$$a(\lambda) < \frac{C_1}{1+\lambda^2}.$$

Moreover, for $|\lambda| \leq 1$, there exists $k_1 > 0$ such that

$$\frac{1+\lambda^2}{1+\lambda^4} < k_1,$$

and so,

$$a(\lambda) < \frac{k_1 C_1}{1+\lambda^2}.$$

Taking $B_1 = \text{máx}\{C_1, k_1C_1\}$ we obtain the estimates. Now assume that $|\lambda| \leq \lambda_0$. Then we observe for any $Q > 0$ that

$$\frac{|\lambda|}{\lambda_0+Q} < 1 < \log(4),$$

which is equivalent to have

$$1 < 2\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}.$$

Using (17) we have that

$$a(\lambda) < \frac{2B_1}{1+\lambda^2}\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}. \qquad \square$$

Now we state and sketch the proof of an important result in **Bourgain**'s work, which we must extend in the case of having variable coefficients.

Lemma 2.3. J. Bourgain (see [1], page 440). *Let u be a sufficiently smooth function in $\mathbb{R}\times I$ such that*

$$\operatorname{supp} u(t)(\cdot) \subseteq [-B, B] \quad \text{for } t \in I. \tag{18}$$

If there are $x_0 \in [-B, B]$ and $t_0 \in I$ such that $u(t_0)(x_0) \neq 0$, then there is $c > 0$ such that for all $Q > 0$ there exists $\lambda > 0$ arbitrarily large such that

$$a(\lambda) > c(a * \cdots * a)(\lambda) \quad \text{and} \quad a(\lambda) > e^{-\frac{\lambda}{Q}}.$$

Proof. We only sketch the proof since many of the estimates have to be extended to the variable coefficients case. The first observation is that

$$(a_0 * a_1 * \cdots * a_k)(\lambda) = \int_{\mathbb{R}^k} a_0(\lambda-\lambda_1-\cdots-\lambda_k)a_1(\lambda_1)\cdots a_k(\lambda_k)d\lambda_1\cdots d\lambda_k. \tag{19}$$

Now we argue by contradiction. Assume that for given $c > 0$, there are $Q > 0$ and λ_0 large enough such that, if $\lambda > \lambda_0 > 0$, then we have either

$$a(\lambda) \leq c(a * \cdots * a)(\lambda) \tag{20}$$

or

$$a(\lambda) \leq e^{-\frac{\lambda}{Q}}. \tag{21}$$

Then it is possible to conclude (see [1]) that a has the global bound for $\lambda \in \mathbb{R}$.

$$a(\lambda) < \frac{2B_1}{1+\lambda^2}\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}} < 2B_1\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}. \tag{22}$$

This fact implies that

$$\left|\widehat{u(t_0)}(\lambda)\right| < 2B_1\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}, \quad \text{for } \lambda \in \mathbb{R}.$$

Using this, it is straightforward to see that

$$u(t_0)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\lambda}\widehat{u(t_0)}(\lambda)d\lambda, \quad x \in \mathbb{R},$$

has an analytic extension in a neighborhood of the real axis. In fact, let $z = z_1 + iz_2 \in \mathbb{C}$ be such that $|z_2| < \frac{1}{2(\lambda_0+Q)}$, then

$$\begin{aligned}
\left|u(t_0)(z_1+iz_2)\right| &\leq (2\pi)^{-1}\int_{\mathbb{R}} e^{-z_2\lambda}\left|\widehat{u(t_0)}(\lambda)\right|d\lambda \\
&\leq B_1\pi^{-1}\int_{\mathbb{R}} e^{|z_2||\lambda|}e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}d\lambda \\
&= B_1\pi^{-1}\int_{\mathbb{R}} e^{-\left(\frac{1}{2(\lambda_0+Q)}-|z_2|\right)|\lambda|}d\lambda \\
&= B_1\pi^{-1}\frac{4(\lambda_0+Q)}{1-2|z_2|(\lambda_0+Q)},
\end{aligned}$$

In other words, the extension

$$u(t_0)(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda z}\widehat{u(t_0)}(\lambda)d\lambda \qquad (23)$$

is well defined in the domain $\Pi = \left\{z \in \mathbb{C} \;:\; |Im\, z| < \frac{1}{2(\lambda_0+Q)}\right\}$.

We claim now that $u(t_0) \equiv 0$. To see this, take $z \in \Pi$ and a sequence $\{z_n\}$ in Π such that $\lim_{n\to\infty} z_n = z$, Then we have that

$$|u(t_0)(z_n) - u(t_0)(z)| \le B\pi^{-1}\int_{\mathbb{R}} \left|e^{i\lambda z_n} - e^{i\lambda z}\right| e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}d\lambda.$$

Now from the generalized Mean Value Theorem, we have for n large enough that

$$\left|e^{i\lambda z_n} - e^{i\lambda z}\right| e^{\frac{-|\lambda|}{2(\lambda_0+Q)}} \le 2|z|^2|\lambda|^2 e^{\frac{-|\lambda|}{2(\lambda_0+Q)}} \in L_1(\mathbb{R}, d\lambda).$$

Then, from the Lebesgue Dominate Convergence Theorem we conclude that

$$\lim_{n\to\infty} |u(t_0)(z_n) - u(t_0)(z)| = 0.$$

In other words, we have shown that $u(t_0)(\cdot)$ is continuous in Π. We will see for any triangle Δ in Π that

$$\int_{\partial\Delta} u(t_0)(z)dz = 0.$$

In fact, from Fubini's Theorem

$$\begin{aligned} 2\pi\int_{\partial\Delta} u(t_0)(z)dz &= \int_{\partial\Delta}\left(\int_{\mathbb{R}} e^{iz\lambda}\widehat{u(t_0)}(\lambda)\, d\lambda\right)dz \\ &= \int_{\mathbb{R}}\left(\int_{\partial\Delta} e^{iz\lambda}dz\right)\widehat{u(t_0)}(\lambda)\, d\lambda. \end{aligned}$$

Since the function $g(z) = e^{iz\lambda}$ is analytic, then the Cauchy Theorem implies that

$$\int_{\partial\Delta} e^{iz\lambda}dz = 0,$$

and so,

$$\int_{\partial\Delta} u(t)(z)dz = 0.$$

We conclude that $u(t_0)(\cdot)$ is an analytic function in Π, by applying Morera's Theorem. Recall that we are assuming that $u(t_0)(x) = 0$ for $x \in [-B, B]$, then we must have that $u(t_0) \equiv 0$ due to its analyticity. This is a contradiction since we are assuming that $u(t_0)(x_0) \neq 0$ for some $x_0 \in [-B, B]$. □

Another clever fact in **J. Bourgain**'s result is related with the continuity of the functions $u^*(\lambda)$ and $|\widehat{u(t)}(\lambda)|$ given by the following result.

Lemma 2.4. ([1], equation 1.12) *Under the hypotheses of Lemma 2.3, there exists $c > 0$ such that for $Q > 0$ there are $t_1 \in I$ and $\lambda \in \mathbb{R}$, with $|\lambda|$ arbitrarily large such that*

$$|\widehat{u(t_1)}(\lambda)| = u^*(\lambda) = a(\lambda) > e^{\frac{-|\lambda|}{Q}}$$

and $a(\lambda) > c(a * \cdots a)(\lambda)$.

A Remark on Bourgain's Results. The key estimative in the proof of Lemma 2.3 in the work by **J. Bourgain** [1] is the global estimate (22), which is obtained arguing by contradiction. In order to extend **J. Bourgain**'s results to the variable coefficients case, we impose some hypotheses in the Fourier transform of the coefficients to obtain similar estimates. The first observation is that global exponential decay (22) holds for $|\lambda|$ bounded (see (16) in Lemma (2.2)). In the coming result, we will assume that the variable coefficient β has a global exponential decay of the form (7), in its Fourier transform with respect to the spatial variable. More concretely, we set the following class of functions

$$\mathcal{A} = \left\{w \in \mathcal{D}'(\mathbb{R}) : e^{|\lambda|}(1+\lambda^2)\widehat{w} \in L^\infty(\mathbb{R})\right\}$$

where $\mathcal{D}'(\mathbb{R})$ denotes the set of distributions in $\mathbb{R}$. We will exhibit below some classes of functions contained in $\mathcal{A}$, which satisfy the global exponential decay (7).

Lemma 2.5. *Let β be a continuous function in $\mathbb{R} \times I$ such that $\beta(t) \in \mathcal{A}$ uniformly for $t \in I$. Let β^* and a_β be defined as in (9) and (10), respectively. Then a_β satisfies the conditions given in Lemma (2.1) and for some $k > 0$*

$$a_\beta(\lambda) \le \frac{ke^{-|\lambda|}}{1+\lambda^2}. \qquad (24)$$

Proof. The result follows by noting that for some $k > 0$, we have for $t \in I$ that

$$|\widehat{\beta(t)}(\lambda)| \le \frac{ke^{-|\lambda|}}{1+\lambda^2}.$$

So, the same estimate holds for a_β. The rest of the proof follows as in Lemma 2.1. □

Now we are in position to establish the extension to the variable coefficients case of the main Lemma in [1].

Lemma 2.6. *Let β_i be continuous functions in $\mathbb{R} \times I$ $(1 \le i \le 5)$ such that $\beta_i(t) \in \mathcal{A}$ uniformly for $t \in I$. Let u be a sufficiently smooth function in $\mathbb{R} \times I$ such that*

$$\operatorname{supp} u(t) \subseteq [-B, B], \quad \text{for all } t \in I.$$

If there are $x_0 \in [-B, B]$ and $t_0 \in I$ such that $u(t_0)(x_0) \neq 0$, then there exists $c > 0$ such that for *all $Q>0$ there is $\lambda>0$ arbitrarily large such that*

$$a(\lambda) > c\big[(a_{\beta_1} * a * \cdots * a)(\lambda) + (a_{\beta_2} * a * \cdots * a)(\lambda) + (a_{\beta_3} * a)(\lambda) + (a_{\beta_4} * a)(\lambda) + (a_{\beta_5} * a)(\lambda)\big]$$

and $a(\lambda) > e^{\frac{-\lambda}{Q}}$.

Proof. For simplicity we only show that

$$a(\lambda) > c\big[(a_{\beta_1} * a * \cdots * a)(\lambda) \, + \, (a_{\beta_2} * a)(\lambda)\big]$$

and $a(\lambda) > e^{\frac{-\lambda}{Q}}$. The general case follows in a similar fashion. We will argue by contradiction, as done by **J. Bourgain** in [1]. Suppose that for $c>0$ that there are $Q>0$ and $\lambda_0>0$ sufficiently large such that, if $\lambda>\lambda_0$, then either

$$a(\lambda) \le c\big[(a_{\beta_1} * a * \cdots * a)(\lambda) \, + \, (a_{\beta_2} * a)(\lambda)\big] \tag{25}$$

or

$$a(\lambda) \le e^{\frac{-\lambda}{Q}}. \tag{26}$$

As shown in Lemma (2.2), there exists $B_1>1$ such that

$$a(\lambda) < \frac{2B_1}{1+\lambda^2}\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}, \quad \text{si} \quad |\lambda| \le \lambda_0. \tag{27}$$

We will see, as in ([1]), that this estimate must be global for $\lambda \in \mathbb{R}$. Assume that (27) were false for $\lambda > \lambda_0$, then there is λ' such that

$$\lambda_0 < \lambda' = \min\Big\{\lambda > \lambda_0 \; : \; a(\lambda) \ge \frac{2B_1}{1+\lambda^2}\, e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}\Big\}.$$

If (26) holds for λ', we will obtain a contradiction. In fact, suppose that (26) holds for λ'. Then from (17), we have that

$$\big|a(\lambda')\big|^2 < \frac{B_1}{1+(\lambda')^4}\, e^{\frac{-\lambda'}{Q}} < \frac{B_1}{1+(\lambda')^4}\, e^{\frac{-\lambda'}{(\lambda_0+Q)}},$$

meaning that

$$a(\lambda') < \left(\frac{B_1}{1+(\lambda')^4}\right)^{1/2} e^{\frac{-\lambda'}{2(\lambda_0+Q)}}.$$

Since we have that $B_1>1$, then

$$\sqrt{B_1}\big(1+(\lambda')^2\big) < 2B_1\big(1+(\lambda')^4\big)^{1/2},$$

then

$$a(\lambda') < \frac{2B_1}{1+(\lambda')^2}\, e^{\frac{-\lambda'}{2(\lambda_0+Q)}},$$

contradicting the definition of λ'.

Now assume that (25) holds for λ' and consider the set

$$U = \Big\{(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k \; : \; |\lambda_1| < \lambda', \, \ldots, \, |\lambda_k| < \lambda'\Big\}.$$

Thus we have that

$$\begin{aligned} a(\lambda') &\le c\int_{\mathbb{R}^k} a_{\beta_1}(\lambda'-\lambda_1-\cdots-\lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\ldots d\lambda_k + c\int_{\mathbb{R}} a_{\beta_2}(\lambda'-\lambda_1)a(\lambda_1)d\lambda_1 \\ &\le c\int_U a_{\beta_1}(\lambda'-\lambda_1-\cdots-\lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\ldots d\lambda_k \\ &\qquad + c\sum_{i=1}^{k}\int\limits_{|\lambda_i|>\lambda'} a_{\beta_1}(\lambda'-\lambda_1-\cdots-\lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\ldots d\lambda_k \\ &\qquad + c\int\limits_{|\lambda_1|<\lambda'} a_{\beta_2}(\lambda'-\lambda_1)a(\lambda_1)d\lambda_1 \; + \; c\int\limits_{|\lambda_1|>\lambda'} a_{\beta_2}(\lambda'-\lambda_1)a(\lambda_1)d\lambda_1. \end{aligned}$$

Now we will see that there is $c_1>0$ such that

$$\int\limits_{|\lambda_1|>\lambda'} a_{\beta_1}(\lambda'-\lambda_1-\cdots-\lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\ldots d\lambda_k \; \le \; c_1 a(\lambda'). \tag{28}$$

From Lemma (2.5), the function a_β satisfies the property (24), and then we have that

$$\int_{-\infty}^{-\lambda'} a_{\beta_1}(\lambda'-\lambda_1-\cdots-\lambda_k)d\lambda_1 \le \; k_1\int_{-\infty}^{-\lambda'} \frac{1}{1+|\lambda'-\lambda_1-\cdots-\lambda_k|^2}d\lambda_1 \le k_1\pi. \tag{29}$$

Similarly we have that

$$\int_{\lambda'}^{\infty} a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)d\lambda_1 \leq k_1\pi. \tag{30}$$

Now from the property (11), we obtain that

$$\begin{aligned}
&\int\limits_{|\lambda_1|>\lambda'} a_{\beta_1}(\lambda' - \lambda_1 \cdots - \lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\dots d\lambda_k \\
&\qquad \leq a(\lambda') \int\limits_{|\lambda_1|>\lambda'} a_{\beta_1}(\lambda' - \lambda_1 \cdots - \lambda_k)a(\lambda_2)\cdots a(\lambda_k)d\lambda_1\dots d\lambda_k \\
&\qquad = a(\lambda')\int_{\mathbb{R}^{k-1}}\left(\int_{-\infty}^{-\lambda'} a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)d\lambda_1\right)a(\lambda_2)\cdots a(\lambda_k)d\lambda_2\dots d\lambda_k \\
&\qquad + a(\lambda')\int_{\mathbb{R}^{k-1}}\left(\int_{\lambda'}^{\infty} a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)d\lambda_1\right)a(\lambda_2)\cdots a(\lambda_k)d\lambda_2\dots d\lambda_k \\
&\qquad \leq 2k_1\pi a(\lambda')\left(\int_{\mathbb{R}} a(\lambda)d\lambda\right)^{k-1},
\end{aligned}$$

obtaining (28) as desired. Arguing in the same way, it is possible to show that there are positive constants $c_2, \dots, c_k, \kappa$ such that for $i = 2, \dots, k$, we have the estimates

$$\int\limits_{|\lambda_1|>\lambda'} a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)\, a(\lambda_1)\cdots a(\lambda_k)\, d\lambda_1\dots d\lambda_k \;\leq\; c_i a(\lambda'),$$

$$\int\limits_{|\lambda_1|>\lambda'} a_{\beta_2}(\lambda' - \lambda_1)a(\lambda_1)\, d\lambda_1 \;\leq\; \kappa a(\lambda').$$

Thus, we get

$$\begin{aligned}
a(\lambda') \leq c\int_U a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\dots d\lambda_k \\
+\; c\int\limits_{|\lambda_1|<\lambda'} a_{\beta_2}(\lambda' - \lambda_1)a(\lambda_1)d\lambda_1 + ca(\lambda')\sum_{i=1}^{k} c_i + c\kappa a(\lambda').
\end{aligned}$$

Then taking c sufficiently small we conclude that

$$\begin{aligned}
a(\lambda') \leq 2c\int_U a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1\dots d\lambda_k \\
+\; 2c\int\limits_{|\lambda_1|<\lambda'} a_{\beta_2}(\lambda' - \lambda_1)a(\lambda_1)d\lambda_1.
\end{aligned}$$

Now, since λ_0 is large, then $2(\lambda_0 + Q) > 1$, and so,

$$-|\lambda' - \lambda_1 - \cdots - \lambda_k| < \frac{-|\lambda' - \lambda_1 - \cdots - \lambda_k|}{2(\lambda_0 + Q)},$$

Again from Lemma (2.5), the function a_{β_1} satisfies the property (24), and then we have that

$$\int_U a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)a(\lambda_1)\cdots a(\lambda_k)d\lambda_1 \ldots d\lambda_k$$

$$\leq k_1(2B_1)^k \int_U \frac{e^{\frac{-|\lambda'-\lambda_1-\cdots-\lambda_k|}{2(\lambda_0+Q)}}}{1+|\lambda'-\lambda_1-\cdots-\lambda_k|^2} \frac{e^{\frac{-|\lambda_1|}{2(\lambda_0+Q)}}}{1+|\lambda_1|^2} \cdots \frac{e^{\frac{-|\lambda_k|}{2(\lambda_0+Q)}}}{1+|\lambda_k|^2} d\lambda_1 \ldots d\lambda_k$$

$$\leq k_1(2B_1)^k e^{\frac{-\lambda'}{2(\lambda_0+Q)}} \int_U \frac{1}{1+|\lambda'-\lambda_1-\cdots-\lambda_k|^2} \frac{1}{1+|\lambda_1|^2} \cdots \frac{1}{1+|\lambda_k|^2} d\lambda_1 \ldots d\lambda_k.$$

Now we consider the sets

$$U_1 = U \cap \Big\{(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k \,:\, \lambda_1 + \cdots + \lambda_k > \frac{\lambda'}{2}\Big\},$$
$$U_2 = U \cap \Big\{(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k \,:\, \lambda_1 + \cdots + \lambda_k < \frac{\lambda'}{2}\Big\},$$

then it is not difficult to show that there exists $\rho_1 > 0$ such that for $i = 1, 2$,

$$\int_{U_i} \frac{1}{1+|\lambda'-\lambda_1-\cdots-\lambda_k|^2} \frac{1}{1+|\lambda_1|^2} \cdots \frac{1}{1+|\lambda_k|^2} d\lambda_1 \ldots d\lambda_k < \frac{\rho_1}{1+(\lambda')^2},$$

Then we also have that

$$\int_U a_{\beta_1}(\lambda' - \lambda_1 - \cdots - \lambda_k)a(\lambda_1)\cdots a(\lambda_k)\, d\lambda_1 \ldots d\lambda_k$$
$$\leq \frac{2k_1\rho_1(2B_1)^k}{1+(\lambda')^2} e^{\frac{-\lambda'}{2(\lambda_0+Q)}}.$$

Similarly, there exists $\rho_2 > 0$ such that

$$\int_{|\lambda_1|<\lambda'} a_{\beta_2}(\lambda' - \lambda_1)a(\lambda_1)d\lambda_1 \leq \frac{2k_2\rho_2(2B_1)}{1+(\lambda')^2} e^{\frac{-\lambda'}{2(\lambda_0+Q)}}.$$

Since c is being taken small enough, the we have that

$$a(\lambda') < \frac{2B_1}{1+(\lambda')^2} e^{\frac{-\lambda'}{2(\lambda_0+Q)}},$$

contradicting again the definition of λ'. In other words, we have the global estimate for $\lambda \in \mathbb{R}$,

$$a(\lambda) < \frac{2B_1}{1+\lambda^2} e^{\frac{-|\lambda|}{2(\lambda_0+Q)}} < 2B_1 e^{\frac{-|\lambda|}{2(\lambda_0+Q)}}.$$

As in the last part of the proof of Lemma 2.3 we also reach a contradiction.

Similarly to the Lemma 2.4, it is possible to show that

Lemma 2.7. *Under the hypotheses of the Lemma 2.6, there exists $c > 0$ such that for given $Q > 0$ there are $t_1 \in I$ and $\lambda \in \mathbb{R}$ arbitrarily large, such that*

$$|\widehat{u(t_1)}(\lambda)| = u^*(\lambda) = a(\lambda) > e^{\frac{-|\lambda|}{Q}}.$$

Moreover, we also have

$$a(\lambda) > c\big[(a_{\beta_1} * a * \cdots * a)(\lambda) + (a_{\beta_2} * a * \cdots * a)(\lambda) + (a_{\beta_3} * a)(\lambda) + (a_{\beta_4} * a)(\lambda) + (a_{\beta_5} * a)(\lambda)\big].$$

2.1. Some Examples of Functions in the Class $\mathcal{A}$.

Now we will exhibit examples of functions in the class $\mathcal{A}$, including functions f defined in $\mathbb{R} \times I$ having the exponential decay (7) uniformly in t.

Example 2.1. Let K and g be functions such that

$$|\widehat{K}(\lambda)| \leq \frac{c_1}{1+|\lambda|^2}, \quad |\widehat{g}(\lambda)| \leq c_2 e^{-|\lambda|}.$$

Then we have that the function $\beta = K * g \in \mathcal{A}$. In fact, a simple computation gives us for some constant $c > 0$ that

$$|\widehat{\beta}(\lambda)| = |\widehat{K * g}(\lambda)| = |\widehat{K}(\lambda)\widehat{g}(\lambda)| \leq \frac{ce^{-|\lambda|}}{1+\lambda^2}.$$

We note that such functions K and g can be even built explicitly. The first observation is that $K(x) = \frac{e^{-|x|}}{2}$ if and only if $\widehat{K}(\lambda) = \frac{1}{1+\lambda^2}$. In fact,

$$\widehat{K}(\lambda) = \frac{1}{2}\int_{\mathbb{R}} e^{-i\lambda x} e^{-|x|} dx$$
$$= \frac{1}{2}\int_{\mathbb{R}} \cos(\lambda x)e^{-|x|}dx - \frac{i}{2}\int_{\mathbb{R}} \sin(\lambda x)e^{-|x|}dx.$$

Since we know that $\sin(\lambda x)e^{-|x|}$ is an odd function (in x), and that $\cos(\lambda x)e^{-|x|}$ is an even function, we have that

$$\widehat{K}(\lambda) = \frac{1}{2}\int_{\mathbb{R}} \cos(\lambda x)e^{-|x|}dx = \int_0^{\infty} \cos(\lambda x)e^{-x}dx.$$

Now, using integration by parts, we conclude that

$$(1+\lambda^2)\int \cos(\lambda x)e^{-x}dx = -\cos(\lambda x)e^{-x} + \lambda \sin(\lambda x)e^{-x},$$

which implies that

$$\begin{aligned}(1+\lambda^2)&\int_0^{\infty} \cos(\lambda x)e^{-x}dx \\ &= -\lim_{t\to\infty} \cos(\lambda t)e^{-t} + 1 + \lambda \lim_{t\to\infty} \sin(\lambda t)e^{-t} = 1.\end{aligned}$$

This means that

$$\widehat{K}(\lambda) = \frac{1}{1+\lambda^2}.$$

Since the Fourier transform is a bijection from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$, we have that $K(x) = \frac{e^{-|x|}}{2}$ if and only if $\widehat{K}(\lambda) = \frac{1}{1+\lambda^2}$. Now, we will build some functions g. Using the same type of estimates, we have for any $\alpha \neq 0$ that $g_\alpha(y) = \frac{1}{\pi}\frac{\alpha}{\alpha^2+y^2}$ if and only if $\widehat{g_\alpha}(\lambda) = e^{-\alpha|\lambda|}$. To see this, we observe that $\widehat{g_\alpha} \in L_2(\mathbb{R})$ and that

$$\begin{aligned} g_\alpha(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x}e^{-\alpha|\lambda|}d\lambda = \frac{1}{\pi}\int_0^{\infty}\cos(\lambda x)e^{-\alpha\lambda}d\lambda \\ &= \frac{1}{\pi}\frac{\alpha}{\alpha^2+y^2},\end{aligned}$$

proceeding as above. Then, for $\alpha \geq 1$, we have that the function $\beta = K * g_\alpha \in \mathcal{A}$, since

$$|\widehat{\beta}(\lambda)| = |\widehat{K * g_\alpha}(\lambda)| = |\widehat{K}(\lambda)\widehat{g_\alpha}(\lambda)| = \frac{e^{-\alpha|\lambda|}}{1+\lambda^2} \leq \frac{e^{-|\lambda|}}{1+\lambda^2}.$$

As we will see below, there are many functions K satisfying the estimate $\widehat{K}(\lambda)| \leq \frac{c_1}{1+|\lambda|^2}$.

Example 2.2. Let ρ be a bounded continuous function such that $1 \leq |\rho(t)|$ for $t \in \mathbb{R}$. Define now the function $G(t)(x) = K_{\rho(t)}(x) = K(\rho(t)x)$, where K is a function as in previous example. Then we have that

$$\widehat{G(t)}(\lambda) = \frac{1}{\rho(t)}\widehat{K}\left(\frac{\lambda}{\rho(t)}\right).$$

Thus the function $\beta(t) = G(t) * g_\alpha \in \mathcal{A}$ uniformly for $t \in \mathbb{R}$. In fact,

$$\begin{aligned}|\widehat{\beta(t)}(\lambda)| &= |\widehat{G(t) * g_\alpha}(\lambda)| = |\widehat{G(t)}(\lambda)\widehat{g_\alpha}(\lambda)| \\ &\leq \frac{\rho(t)e^{-\alpha|\lambda|}}{\rho^2(t)+\lambda^2} \leq \frac{Me^{-|\lambda|}}{1+\lambda^2}.\end{aligned}$$

Example 2.3. Let K be a continuous function such that $K, \partial_x^2 K \in L^1(\mathbb{R})$. Then we have that

$$\begin{aligned}(1+\lambda^2)|\widehat{K}(\lambda)| &= |\widehat{K}(\lambda)| + |\widehat{\partial_x^2 K}(\lambda)| \\ &\leq \int_{\mathbb{R}} |K(x)|dx + \int |\partial_x^2 K(x)|dx.\end{aligned}$$

Thus we conclude that

$$|\widehat{K}(\lambda)| \leq \frac{c}{1+\lambda^2},$$

providing in addition functions K as needed in previous examples. Now define $h(x) = e^{-\frac{1}{4}x^2}$. Then for some positive constant k_1, we have that $\widehat{h}(\lambda) = k_1 e^{-\lambda^2}$. As a consequence of this, there is a positive constant k_2 such that for $\lambda \in \mathbb{R}$

$$|\widehat{h}(\lambda)| \leq k_2 e^{-|\lambda|}.$$

Then we have that $\beta = K * h$ and $\beta(t) = K_{\rho(t)} * h$ belong to $\mathcal{A}$, uniformly in $t \in \mathbb{R}$, for any function ρ as in the second example.

3. Unique Continuation Results

In this section we present some results related with unique continuation for the differential equation (6), including the extension of **J. Bourgain** results in [1] to the variable coefficients case. We start the discussion by proving an important property of the function a, whose proof was not included in **J. Bourgain**'s paper ([1]).

Lemma 3.1. *Let $\lambda_1 \in \mathbb{R}$, u be as in Lemma 2.1 and $\lambda \in \mathbb{R}$ be such that $a(\lambda) < 1$. Then there exist a constant $\rho > 1$ independent of λ_1 and λ such that for $\overline{\lambda} = \min\{|\lambda_1|, |\lambda|\}$ we have the estimate*

$$a(\overline{\lambda})\big[1 + |\log a(\overline{\lambda})|\big] \leq \rho\big[a(\lambda) + a(\lambda_1)\big]\big[1 + |\log a(\lambda)|\big]. \tag{31}$$

Proof. If $\overline{\lambda} = |\lambda|$, then the estimate follows since a is a nonnegative even function. Assume that $\overline{\lambda} = |\lambda_1|$. If $|\lambda_1| \leq |\lambda|$, then $a(\lambda) \leq a(\lambda_1)$. Thus, $a(\lambda_1) \leq 1$ implies that

$$\big|\log a(\lambda_1)\big| \leq \big|\log a(\lambda)\big|,$$

and so we have the estimate. Now, recall that a is a bounded function, thus, for $a(\lambda_1) > 1$, there exists $C > 0$ such that

$$a(\lambda) \leq 1 < a(\lambda_1) < C.$$

Taking $k_1 < 1$ and such that $k_1 C < 1$, we obtain $k_1 a(\lambda_1) < 1$. Thus we have either

$$k_1 a(\lambda_1) \leq a(\lambda) < 1 \quad \text{or} \quad a(\lambda) \leq k_1 a(\lambda_1) < 1.$$

In the first case, we use that $f(x) = x\big(1 + |\log x|\big)$, $x > 0$, is an increasing function to conclude that

$$\begin{aligned} a(\lambda_1)\big[1 + |\log a(\lambda_1)|\big] &\le k_1^{-1} a(\lambda)\big[1 + |\log\big(k_1^{-1} a(\lambda)\big)|\big] \\ &\le k_1^{-1}\big[a(\lambda) + a(\lambda_1)\big]\big[1 + |\log k_1| + |\log a(\lambda)|\big] \\ &\le k_1^{-1}\big(1 + |\log k_1|\big)\big[a(\lambda) + a(\lambda_1)\big]\big[1 + |\log a(\lambda)|\big]. \end{aligned}$$

In the second case, if $a(\lambda) \le k_1 a(\lambda_1) < 1$, then $\big|\log\big(k_1 a(\lambda_1)\big)\big| \le |\log a(\lambda)|$. Using this we obtain that

$$\begin{aligned} &a(\lambda_1)\big[1 + |\log a(\lambda_1)|\big] \\ &\quad\le a(\lambda_1)\big[1 + |\log k_1| + |\log\big(k_1 a(\lambda_1)\big)|\big] \\ &\quad\le \big(1 + |\log k_1|\big)\big[a(\lambda) + a(\lambda_1)\big]\big[1 + |\log a(\lambda)|\big]. \ \square \end{aligned}$$

As we mention in the introduction, we need to apply Theorem (1.3) to be able to handle the derivative of u.

Lemma 3.2. *Let u be as in the Paley–Wiener Theorem. Let $\lambda_1 > 0$ and $t_0 \in I$ be fixed, and let $\widehat{u(t_0)}(z)$ be the analytic extension of the Fourier transform of $u(t_0)(x)$. Then for $\sigma \in \mathbb{R}$ such that $|\sigma|$ is small enough, we have that*

$$\sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi + i\sigma)| \le 2 \sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi)|. \tag{32}$$

Proof. Let $M > 0$ be such that

$$|u(x,t)| < M, \quad x \in [-B, B]. \tag{33}$$

Now take $\xi \in \mathbb{R}$ such that $|\xi| \ge \lambda_1$, then

$$\begin{aligned} |\widehat{u(t_0)}(\xi + i\sigma)| - |\widehat{u(t_0)}(\xi)| &\le \bigg|\int_{-B}^{B} e^{-i(\xi+i\sigma)x} u(t_0)(x)dx \\ &\quad - \int_{-B}^{B} e^{-i\xi x} u(t_0)(x)dx\bigg| \\ &\le \int_{-B}^{B} \big|e^{\sigma x} - 1\big|\big|u(t_0)(x)\big| \le M\int_{-B}^{B}\big|e^{\sigma x} - 1\big|dx \\ &\le M\left(\frac{e^{|\sigma|B} + e^{-|\sigma|B} - 2}{|\sigma|}\right). \end{aligned}$$

But we have that

$$\lim_{|\sigma|\to 0} \frac{e^{|\sigma|B} + e^{-|\sigma|B} - 2}{|\sigma|} = 0,$$

then we are able to take $|\sigma|$ small enough such that

$$M\left(\frac{e^{|\sigma|B} + e^{-|\sigma|B} - 2}{|\sigma|}\right) \le \sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi)|.$$

Thus we have that

$$|\widehat{u(t_0)}(\xi + i\sigma)| \le 2 \sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi)|,$$

obtaining the desired estimate (32). $\square$

Using this result, we also have that

Lemma 3.3. *Let u be a sufficiently smooth function in $\mathbb{R} \times I$ such that*

$$\operatorname{supp} u(t) \subseteq [-B, B], \quad \text{for } t \in I.$$

Then we can choose $\lambda_1 \in \mathbb{R}$, with $|\lambda_1|$ sufficiently large, such that for

$$|\sigma| < B^{-1}\big[1 + |\log a(\lambda_1)|\big]^{-1},$$

we have that

$$\sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi + i\sigma)| \le 2 \sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi)|,$$

where $t_0 \in I$ and $\widehat{u(t_0)}(z)$ denotes the analytic extension of the Fourier transform of $u(t_0)(x)$.

Proof. Since $\text{lím}_{|\lambda|\to\infty} a(\lambda) = 0$, then we can choose $|\lambda_1|$ sufficiently large such that

$$|\sigma| < B^{-1}\big[1 + |\log a(\lambda_1)|\big]^{-1}$$

is small enough. Thus applying Lemma 3.2 we have that

$$\sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi + i\sigma)| \le 2 \sup_{|\xi|\ge\lambda_1} |\widehat{u(t_0)}(\xi)|. \qquad \square$$

Now we present one of the main theorems in this work.

Theorem 3.1. *Let r_i and $\partial_x r_j$ be continuous functions in $\mathbb{R} \times I$ for $i = 1, 2, 3$ and $j = 1, 2$ with $r_i, \partial_x r_j \in L^1(\mathbb{R})$ uniformly for $t \in I$, and such that $\partial_x r_j(t), r_3(t) \in \mathcal{A}$ uniformly for $t \in I$. If u is a sufficiently smooth solution of the (KdV) equation with variable coefficients*

$$u_t + \gamma\partial_x^3 u + k r_1(t)(x) u^{k-1}\partial_x u + r_2(t)(x)\partial_x u + r_3(t)(x)u = 0 \tag{34}$$

in $\mathbb{R} \times I$, $(k \ge 2)$ such that

$$\operatorname{supp} u(t) \subseteq [-B, B], \quad \text{for } t \in I.$$

Then $u(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times I$.

Proof. Hereafter we set

$$\begin{aligned} F(t)(x) &= F(t)(x, r_1, r_2, r_3, u, \partial_x u) \\ &= -(k r_1(t)(x) u^{k-1}\partial_x u + r_2(t)(x)\partial_x u + r_3(t)(x)u) \\ &= -\big(r_1(t)(x)\partial_x(u^k) + r_2(t)(x)\partial_x u + r_3(t)(x)u\big). \end{aligned}$$

We first note that $\widehat{r_1(t)}$ and $\widehat{r_2(t)}$ also have the global decay estimate (7), uniformly in t. In other words, $r_1(t), r_2(t) \in \mathcal{A}$, uniformly in t. In fact, using that $r_i(t) \in L^1(\mathbb{R})$ uniformly in t, we have for $i = 1, 2$, that

$$|\widehat{r_i(t)}(\lambda)| < C.$$

Now, since the function $f(\lambda) = \frac{e^{-|\lambda|}}{1+\lambda^2}$ is continuous and is never zero in $[-1,1]$, there are positive constants k_i such that

$$|\widehat{r_i(t)}(\lambda)| \leq \frac{k_i e^{-|\lambda|}}{1+\lambda^2}, \quad \lambda \in [-1,1].$$

But $\partial_x r_i(t)$ satisfies condition (7) uniformly in t, implying for $|\lambda| > 1$ that

$$|\widehat{r_i(t)}(\lambda)| \leq \frac{c_i e^{-|\lambda|}}{|\lambda|(1+\lambda^2)} \leq \frac{c_i e^{-|\lambda|}}{1+\lambda^2}.$$

Thus we conclude that there are positive constants c_i such that for $t \in I$ and $\lambda \in \mathbb{R}$,

$$|\widehat{r_i(t)}(\lambda)| \leq \frac{c_i e^{-|\lambda|}}{1+\lambda^2}.$$

Now we will argue by contradiction. In other words, assume that there are $x_0 \in [-B,B]$ and $t_0 \in I$ such that $u(t_0)(x_0) \neq 0$. Then, from the Paley–Wiener Theorem we know that $\widehat{u(t)}$ has an analytic extension in $\mathbb{C}$ and there is $\kappa_1 > 0$ such that for $t \in I$,

$$|\widehat{u(t)}(\lambda + i\sigma)| \leq \kappa_1 e^{|\sigma|B}, \quad \lambda, \sigma \in \mathbb{R}. \tag{35}$$

As in the introduction, using the semigroup associated to the linear equation

$$u_t + \gamma u_{xxx} = 0,$$

we know that Duhamel´s principle implies for $t_1 \in I$ that the solution of (34) can be expressed as

$$u(t)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\lambda x + \gamma\lambda^3(t-t_1))}\widehat{u(t_1)}(\lambda)d\lambda + \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{t_1}^{t} e^{i(\lambda x+\gamma\lambda^3(t-\tau))}\widehat{F(\tau)}(\lambda)d\tau\right)d\lambda.$$

This for fixed $t_1,\ t_2 \in I$ we have that

$$u(t_2)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\lambda x + \gamma\lambda^3(t_2-t_1))}\widehat{u(t_1)}(\lambda)d\lambda + \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{t_1}^{t_2} e^{i(\lambda x+\gamma\lambda^3(t_2-\tau))}\widehat{F(\tau)}(\lambda)d\tau\right)d\lambda.$$

Moreover, in terms of the Fourier transform

$$\widehat{u(t_2)}(\lambda) = e^{i\gamma\lambda^3\Delta t}\left[\widehat{u(t_1)}(\lambda) + \int_0^{\Delta t} e^{-i\gamma\lambda^3\tau}\widehat{F(\tau+t_1)}(\lambda)d\tau\right],$$

where $\Delta t = t_2 - t_1$. We also have that the analytic extension of $\widehat{u(t_2)}(\lambda)$, for $\lambda, \sigma \in \mathbb{R}$, satisfies that

$$\widehat{u(t_2)}(\lambda+i\sigma) = e^{i\gamma(\lambda+i\sigma)^3\Delta t}\left[\widehat{u(t_1)}(\lambda+i\sigma) + \int_0^{\Delta t} e^{-i\gamma(\lambda+i\sigma)^3\tau}\widehat{F(\tau+t_1)}(\lambda+i\sigma)d\tau\right],$$

since we also have that $\widehat{F(t)}$ admits an analytic extension in $\mathbb{C}$ ($F(t)$ has compact support). Using that

$$(\lambda+i\sigma)^3 = (\lambda^3 - 3\lambda\sigma^2) + (3\sigma\lambda^2 - \sigma^3)i,$$

for $\lambda, \sigma \in \mathbb{R}$ and $t_1, t_2 \in I$, we conclude that

$$\begin{aligned}|\widehat{u(t_2)}(\lambda+i\sigma)| &= e^{\gamma(\sigma^3-3\sigma\lambda^2)\Delta t}\left|\widehat{u(t_1)}(\lambda+i\sigma) + \int_0^{\Delta t} e^{-i\gamma(\lambda+i\sigma)^3\tau}\widehat{F(\tau+t_1)}(\lambda+i\sigma)d\tau\right| \\ &\geq e^{\gamma(\sigma^3-3\sigma\lambda^2)\Delta t}\left[\left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - \int_0^{\Delta t} e^{\gamma(3\sigma\lambda^2-\sigma^3)\tau}\left|\widehat{F(\tau+t_1)}(\lambda+i\sigma)\right|d\tau\right].\end{aligned}$$

Suppose that $\lambda \in \mathbb{R}$ and that σ is such that $\gamma\sigma\Delta t < 0$ with $|\sigma| < 1$ then

$$\gamma(\sigma^3 - 3\sigma\lambda^2)\Delta t = (3\lambda^2 - \sigma^2)|\sigma\gamma\Delta t|.$$

Using (35), we conclude for $\kappa = \kappa_1 e^B$ that

$$\begin{aligned}\kappa > \kappa_1 e^{|\sigma|B} &\geq e^{-\sigma^2|\gamma\sigma\Delta t|}e^{3\lambda^2|\gamma\sigma\Delta t|}\left[\left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - \int_0^{\Delta t} e^{3\gamma\lambda^2\sigma\tau}e^{-\gamma\sigma^3\tau}\left|\widehat{F(t_1+\tau)}(\lambda+i\sigma)\right|d\tau\right] \\ &\geq e^{-\sigma^2|\gamma\sigma\Delta t|}e^{3\lambda^2|\gamma\sigma\Delta t|}\left[\left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - A\int_0^{\Delta t} e^{-\lambda^2|3\gamma\sigma\tau|}\left|\widehat{F(t_1+\tau)}(\lambda+i\sigma)\right|d\tau\right],\end{aligned}$$

where $A = e^{2|\gamma|T}$. Observe for $\Delta t > 0$ that

$$\int_0^{\Delta t} e^{-3\lambda^2|\gamma\sigma\tau|}\left|\widehat{F(t_1+\tau)}(\lambda+i\sigma)\right|d\tau = \int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{F(t_1+\tau)}(\lambda+i\sigma)\right|d\tau, \tag{36}$$

and for $\Delta t < 0$,

$$\int_0^{\Delta t} e^{-3\sigma^2|\gamma\sigma\tau|}\left|\widehat{F(t_1+\tau)}(\lambda+i\sigma)\right|d\tau = -\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{F(t_1-\tau)}(\lambda+i\sigma)\right|d\tau. \tag{37}$$

Then without loss of generality we can use either (36) or (37). Thus for $t_1, t_2 \in I$ and $A = e^{2|\gamma|T}$, we have that

$$\kappa e^{\sigma^2|\gamma\sigma\Delta t|} > e^{3\lambda^2|\gamma\sigma\Delta t|}\left[\left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - A\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{F(t_1+\tau)}(\lambda+i\sigma)\right|d\tau\right]. \tag{38}$$

But we know that

$$(\widehat{f\partial_x g})(\lambda) = i\lambda(\widehat{fg})(\lambda) - (\widehat{\partial_x f g})(\lambda), \quad \text{and} \quad \left(\widehat{f_1\cdots f_k}\right)(\lambda) = \left(\widehat{f_1} * \cdots * \widehat{f_k}\right)(\lambda).$$

Then for any $t \in I$,

$$\begin{aligned}\widehat{F(t)}(\lambda) = &-i\lambda\left(\widehat{r_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\right)(\lambda) + \left(\widehat{\partial_x r_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\right)(\lambda)\\ &- i\lambda\left(\widehat{r_2(t)} * \widehat{u(t)}\right)(\lambda) + \left(\widehat{\partial_x r_2(t)} * \widehat{u(t)}\right)(\lambda) - \left(\widehat{r_3(t)} * \widehat{u(t)}\right)(\lambda)\end{aligned}$$

Since $|\sigma| < 1$, then for $\|\lambda\| > 1$ we have that $|\lambda + i\sigma| < 2|\lambda|$. Using this

$$\begin{aligned}\left|\widehat{F(t)}(\lambda+i\sigma)\right| \leq\ &|\lambda+i\sigma|\left|\left(\widehat{r_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\right)(\lambda+i\sigma)\right| +\\ &\left|\left(\widehat{\partial_x r_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\right)(\lambda+i\sigma)\right| + |\lambda+i\sigma|\left|\left(\widehat{r_2(t)} * \widehat{u(t)}\right)(\lambda+i\sigma)\right| +\\ &\left|\left(\widehat{\partial_x r_2(t)} * \widehat{u(t)}\right)(\lambda+i\sigma)\right| + \left|\left(\widehat{r_3(t)} * \widehat{u(t)}\right)(\lambda+i\sigma)\right|\\ <\ &2|\lambda|\widehat{G(t)}(\lambda+i\sigma),\end{aligned}$$

where G is defined by

$$\begin{aligned}\widehat{G(t)}(z) = &\left|\left(\widehat{r_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\right)(z)\right| + \left|\left(\widehat{\partial_x r_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\right)(z)\right|\\ &+ \left|\left(\widehat{r_2(t)} * \widehat{u(t)}\right)(z)\right| + \left|\left(\widehat{\partial_x r_2(t)} * \widehat{u(t)}\right)(z)\right| + \left|\left(\widehat{r_3(t)} * \widehat{u(t)}\right)(z)\right|.\end{aligned}$$

For convenience we set $\beta_1 = r_1$, $\beta_2 = \partial_x r_1$, $\beta_3 = r_2$, $\beta_4 = \partial_x r_2$, and $\beta_5 = r_3$. Then for $t_1, t_2 \in I$, $|\lambda| > 1$, $|\sigma| < 1$ we have that

$$\kappa e^{\sigma^2|\gamma\sigma\Delta t|} > e^{3\lambda^2|\gamma\sigma\Delta t|}\left[\left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{G(t_1+\tau)}(\lambda+i\sigma)\right|d\tau\right]. \tag{39}$$

Note now that

$$\begin{aligned}&\left|\widehat{u(t_1)}(\lambda+i\sigma)\right| - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma\tau|}\left|\widehat{G(t_1+\tau)}(\lambda+i\sigma)\right|d\tau\\ &\qquad\geq \left|\widehat{u(t_1)}(\lambda)\right| - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{G(t_1+\tau)}(\lambda)\right|d\tau\\ &\qquad\qquad - \left|\widehat{u(t_1)}(\lambda+i\sigma) - \widehat{u(t_1)}(\lambda)\right|\\ &\qquad - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{G(t_1+\tau)}(\lambda+i\sigma) - \widehat{G(t_1+\tau)}(\lambda)\right|d\tau.\end{aligned}$$

Then choosing c, t_1 and λ ($|\lambda|$ large enough) as in Lemma 2.7, with $Q > 0$, which will be specified below, and $\sigma = \sigma(\lambda)$ such that

$$1 - \frac{2A}{3c|\gamma\sigma\lambda|} > \frac{1}{2}, \tag{40}$$

we have that

$$
\begin{aligned}
&\left|\widehat{u(t_1)}(\lambda)\right| - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{G(t_1+\tau)}(\lambda)\right|d\tau \\
&= a(\lambda) - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\Big[\Big|\Big(\widehat{\beta_1(t_1+\tau)} * \widehat{u(t_1+\tau)} * \cdots * \widehat{u(t_1+\tau)}\Big)(\lambda)\Big| \\
&\qquad + \Big|\Big(\widehat{\beta_2(t_1+\tau)} * \widehat{u(t_1+\tau)} * \cdots * \widehat{u(t_1+\tau)}\Big)(\lambda)\Big| \\
&\qquad + \Big|\Big(\widehat{\beta_3(t_1+\tau)} * \widehat{u(t_1+\tau)}\Big)(\lambda)\Big| + \Big|\Big(\widehat{\beta_4(t_1+\tau)} * \widehat{u(t_1+\tau)}\Big)(\lambda)\Big| \\
&\qquad + \Big|\Big(\widehat{\beta_5(t_1+\tau)} * \widehat{u(t_1+\tau)}\Big)(\lambda)\Big|\Big]d\tau \\
&\geq a(\lambda) - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\Big[(a_{\beta_1} * a * \cdots * a)(\lambda) + (a_{\beta_2} * a * \cdots * a)(\lambda) \\
&\qquad + (a_{\beta_3} * a)(\lambda) + (a_{\beta_4} * a)(\lambda) + (a_{\beta_5} * a)(\lambda)\Big]d\tau \\
&> a(\lambda) - 2Ac^{-1}|\lambda|a(\lambda)\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}d\tau \\
&= a(\lambda) - \frac{2A|\lambda|}{3c|\gamma\sigma|\lambda^2}\Big(1 - e^{-3\lambda^2|\gamma\sigma\Delta t|}\Big)a(\lambda) > a(\lambda) - \frac{2A}{3c|\gamma\sigma\lambda|}a(\lambda) > \frac{1}{2}a(\lambda).
\end{aligned} \tag{41}
$$

From the generalized Mean Value Theorem, there is $\sigma_0 \in \mathbb{R}$, $|\sigma_0| < |\sigma|$, such that

$$
\begin{aligned}
\left|\widehat{u(t_1)}(\lambda + i\sigma) - \widehat{u(t_1)}(\lambda)\right| &\leq |\sigma|\left|\big(\widehat{u(t_1)}\big)'(\lambda + i\sigma_0)\right| \\
&\leq |\sigma| \sup_{|\xi|\geq|\lambda|}\left|\big(\widehat{u(t_1)}\big)'(\xi + i\sigma_0)\right|
\end{aligned}
$$

then if μ and ρ are as in Theorem 1.3 and in Lemma 3.1 respectively, and σ is such that

$$
\begin{aligned}
|\sigma| &< (8\mu\rho B)^{-1}(1 + |\log a(\lambda)|)^{-1} \\
&< (8\mu B)^{-1}(1 + |\log a(\lambda)|)^{-1} \\
&< B^{-1}(1 + |\log a(\lambda)|)^{-1} < 1,
\end{aligned}
$$

we conclude (using Lemma 3.3, Theorem 1.3 and recalling that $|\lambda|$ is large enough) that

$$
\begin{aligned}
&\left|\widehat{u(t_1)}(\lambda + i\sigma) - \widehat{u(t_1)}(\lambda)\right| \leq \mu B|\sigma|\Big(\sup_{|\xi|\geq|\lambda|}|\widehat{u(t_1)}(\xi)|\Big) \times \\
&\qquad \Big[1 + \big|\log\big(\sup_{|\xi|\geq|\lambda|}|\widehat{u(t_1)}(\xi)|\big)\big|\Big] \\
&\qquad = \mu B|\sigma|a(\lambda)\big[1 + |\log a(\lambda)|\big] < \frac{1}{8}a(\lambda).
\end{aligned} \tag{42}
$$

We now claim that the conditions imposed to λ and $\sigma(\lambda)$ are verifiable. If $A_1 = \frac{2A}{3|\gamma|c}$, then we will see that

$$1 - \frac{A_1}{|\sigma\lambda|} > \frac{1}{2}$$

which is equivalent to have

$$\frac{2A_1}{|\lambda|} < |\sigma|, \tag{43}$$

then we can choose λ such that $\frac{2A_1}{|\lambda|} < 1$. But we also must have that

$$|\sigma| < (8\mu\rho B)^{-1}(1 + |\log a(\lambda)|)^{-1} < 1,$$

meaning that λ must be such that

$$\frac{A_2}{|\lambda|} < \frac{1}{1 + |\log a(\lambda)|} \text{ and } \frac{1}{B(1 + |\log a(\lambda)|)} < 1, \tag{44}$$

where $A_2 = 16A_1\mu\rho B$. The second inequality is reached by choosing λ large enough, since $\text{lím}_{\lambda\to\infty} a(\lambda) = 0$. For the first inequality, we must recall that given $Q > 0$ we can take λ such that

$$a(\lambda) > e^{-\frac{|\lambda|}{Q}}.$$

As a consequence of this , if we take Q such that $\frac{2}{Q} < \frac{1}{A_2}$ and $|\lambda| > Q$ we conclude that

$$|\log a(\lambda)| < \frac{|\lambda|}{Q},$$

implying that

$$1+|\log a(\lambda)| < \frac{2|\lambda|}{Q} < \frac{|\lambda|}{A_2},$$

and obtaining the desired condition. Now we will establish an estimate for the expression

$$2A|\lambda|\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\left|\widehat{G(t_1+\tau)}(\lambda+i\sigma)-\widehat{G(t_1+\tau)}(\lambda)\right|d\tau.$$

For $t\in I$, we have that

$$\begin{aligned}
&\left|\left(\widehat{\beta_1(t)}*\widehat{u(t)}*\cdots*\widehat{u(t)}\right)(\lambda+i\sigma)-\left(\widehat{\beta_1(t)}*\widehat{u(t)}*\cdots*\widehat{u(t)}\right)(\lambda)\right|\\
&=\left|\int_{\mathbb{R}^k}\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma)\widehat{u(t)}(\lambda_1)\cdots\widehat{u(t)}(\lambda_{k-1})\widehat{\beta_1(t)}(\lambda_k)d\lambda_1\ldots d\lambda_k\right.\\
&\qquad\left.-\int_{\mathbb{R}^k}\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k)\widehat{u(t)}(\lambda_1)\cdots\widehat{u(t)}(\lambda_{k-1})\widehat{\beta_1(t)}(\lambda_k)d\lambda_1\ldots d\lambda_k\right|\\
&\le\int_{\mathbb{R}^k}\left|\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma)-\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k)\right|\left|\widehat{u(t)}(\lambda_1)\right|\\
&\qquad\cdots\left|\widehat{u(t)}(\lambda_{k-1})\right|\left|\widehat{\beta_1(t)}(\lambda_k)\right|d\lambda_1\ldots d\lambda_k\\
&\le\int_{\mathbb{R}^k}\left|\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma)-\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k)\right|a(\lambda_1)\\
&\qquad\cdots a(\lambda_{k-1})a_{\beta_1}(\lambda_k)d\lambda_1\ldots d\lambda_k.
\end{aligned}$$

From the generalized Mean Value Theorem, there is $\sigma_1\in\mathbb{R}$ with $|\sigma_1|<|\sigma|$ such that

$$\left|\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma)-\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k)\right|\le|\sigma|\left|\left(\widehat{u(t)}\right)'(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma_0)\right|.$$

Now, taking $\overline{\lambda}=\text{mín}\{|\lambda|,|\lambda-\lambda_1-\cdots-\lambda_k|\}$, and using Theorem 1.3, we conclude that

$$\begin{aligned}
&\left|\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma)-\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k)\right|\le|\sigma|\sup_{|\xi|\ge|\lambda-\lambda_1-\cdots-\lambda_k|}\left|\left(\widehat{u(t)}\right)'(\xi+i\sigma_0)\right|\\
&\qquad\le|\sigma|\sup_{|\xi|\ge\overline{\lambda}}\left|\left(\widehat{u(t)}\right)'(\xi+i\sigma_0)\right|\le\mu B|\sigma|\Big(\sup_{|\xi|\ge\overline{\lambda}}|\widehat{u(t)}(\xi)|\Big)\Big[1+\Big|\log\big(\sup_{|\xi|\ge\overline{\lambda}}|\widehat{u(t)}(\xi)|\big)\Big|\Big]\\
&\qquad\le\mu B|\sigma|\Big(\sup_{|\xi|\ge\overline{\lambda}}\sup_{t\in I}|\widehat{u(t)}(\xi)|\Big)\Big[1+\Big|\log\big(\sup_{|\xi|\ge\overline{\lambda}}\sup_{t\in I}|\widehat{u(t)}(\xi)|\big)\Big|\Big]\\
&\qquad\le\mu B|\sigma|a(\overline{\lambda})\big(1+|\log a(\overline{\lambda})|\big).
\end{aligned}$$

Then using the estimate (31) in Lemma 3.1, we obtain that

$$\begin{aligned}
&\left|\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k+i\sigma)-\widehat{u(t)}(\lambda-\lambda_1-\cdots-\lambda_k)\right|\\
&\qquad\le\mu\rho B|\sigma|\big(a(\lambda)+a(\lambda-\lambda_1-\cdots-\lambda_k)\big)\big(1+|\log a(\lambda)|\big)\\
&\qquad\le a(\lambda)+a(\lambda-\lambda_1-\cdots-\lambda_k).
\end{aligned}$$

As a consequence of this, for $t \in I$,

$$\begin{aligned}\Big|\Big(\widehat{\beta_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\Big)(\lambda + i\sigma) - \Big(\widehat{\beta_1(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\Big)(\lambda)\Big| \\ \leq \int_{\mathbb{R}^k} \big[a(\lambda) + a(\lambda - \lambda_1 - \cdots - \lambda_k)\big] a(\lambda_1) \cdots a(\lambda_{k-1}) a_{\beta_1}(\lambda_k) d\lambda_1 \cdots d\lambda_k \\ = C_1 a(\lambda) + (a_{\beta_1} * a * \cdots * a)(\lambda),\end{aligned}$$

where the constant C_1 is taken as

$$C_1 = \left(\int_{\mathbb{R}} a_{\beta_1}(\varsigma) d\varsigma\right) \left(\int_{\mathbb{R}} a(\varsigma) d\varsigma\right)^{k-1}.$$

In a similar fashion, it can be shown the existence of positive constants C_2, C_3, C_4 and C_5 such that for $j = 3, 4, 5$

$$\Big|\Big(\widehat{\beta_2(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\Big)(\lambda + i\sigma) - \Big(\widehat{\beta_2(t)} * \widehat{u(t)} * \cdots * \widehat{u(t)}\Big)(\lambda)\Big| \leq C_2 a(\lambda) + (a_{\beta_2} * a * \cdots * a)(\lambda),$$

$$\Big|\Big(\widehat{\beta_j(t)} * \widehat{u(t)}\Big)(\lambda + i\sigma) - \Big(\widehat{\beta_j(t)} * \widehat{u(t)}\Big)(\lambda)\Big| \leq C_j a(\lambda) + (a_{\beta_j} * a)(\lambda).$$

These facts imply that there is a positive constant $C > 0$ such that

$$\begin{aligned}\big|\widehat{G(t_1+\tau)}(\lambda + i\sigma) - \widehat{G(t_1+\tau)}(\lambda)\big| \leq C a(\lambda) + (a_{\beta_1} * a * \cdots * a)(\lambda) \\ + (a_{\beta_2} * a * \cdots * a)(\lambda) + (a_{\beta_3} * a)(\lambda) + (a_{\beta_4} * a)(\lambda) + (a_{\beta_5} * a)(\lambda).\end{aligned}$$

Again, from Lemma 2.7, we have that

$$\begin{aligned}2A \int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau} \big|\widehat{G(t_1+\tau)}(\lambda + i\sigma) - \widehat{G(t_1+\tau)}(\lambda)\big| d\tau \leq 2A(C + c^{-1}) a(\lambda) \int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau} d\tau \\ \leq \frac{2A(C + c^{-1})}{3|\gamma\sigma\lambda^2|} a(\lambda) < \frac{a(\lambda)}{8|\lambda|},\end{aligned} \tag{45}$$

taking λ and σ such that

$$\frac{16A(C + c^{-1})}{3|\gamma\lambda|} < |\sigma| < 1.$$

Now, arguing as in (43) and (44), we can choose λ and $\sigma(\lambda)$ with this condition satisfying

$$\frac{1}{3|\gamma\lambda|} < |\sigma|. \tag{46}$$

Thus, by choosing t_1, λ and $\sigma(\lambda)$, and also with (39), (41), (42) and (45), we have that for some positive constant κ that

$$\kappa e^{\sigma^2|\gamma\sigma\Delta t|} > e^{3\lambda^2|\gamma\sigma\Delta t|}\Big[\frac{1}{2}a(\lambda) - \frac{1}{8}a(\lambda) - \frac{1}{8}a(\lambda)\Big].$$

In other words, we have shown that

$$a(\lambda) < 4\kappa e^{\sigma^2|\gamma\sigma\Delta t|} e^{-3\lambda^2|\gamma\sigma\Delta t|}.$$

Thus choosing $t_2 \in I$ such that $T \leq |\Delta t| \leq 2T$ and using Lemma 2.7 and condition (46) we conclude that

$$e^{\frac{-|\lambda|}{Q}} < a(\lambda) < 4\kappa e^{2|\gamma|T} e^{-\lambda T},$$

which is equivalent to have

$$e^{|\lambda|T} < A_3 e^{\frac{|\lambda|}{Q}},$$

with $A_3 = 4\kappa e^{2|\gamma|T}$. But this is a contradiction taking $Q(T)$ large enough.

The first consequence of the proof of previous result is the extension of the unique continuation result due to **J. Bourgain** for equation (34) for variable coefficients depending only on t. The proof does not require imposing any decay condition on the coefficients.

Theorem 3.2. *Let r_i $(i = 1, 2, 3)$ be continuous functions in I. If u is a sufficiently smooth solution of the (KdV) equation with variable coefficients*

$$u_t + \gamma\partial_x^3 u + k r_1(t) u^{k-1}\partial_x u + r_2(t)\partial_x u + r_3(t) u = 0, \tag{47}$$

such that

$$\operatorname{supp} u(t)(\cdot) \subseteq [-B, B], \quad \textit{for } t \in I.$$

then $u(x,t) = 0$ for $(x,t) \in \mathbb{R} \times I$.

Proof. Let F be defined as

$$\begin{aligned} F(t)(x) &= F(t)(x, r_1, r_2, r_3, u, \partial_x u) \\ &= -(k r_1(t) u^{k-1}\partial_x u + r_2(t)\partial_x u + r_3(t) u) \\ &= -(r_1(t)\partial_x(u^k) + r_2(t)\partial_x u + r_3(t) u). \end{aligned}$$

As in Theorem 3.1, there is $\kappa > 0$ such that for $t_1, t_2 \in I$, $\Delta t = t_2 - t_1$, $\gamma\sigma\Delta t < 0$ and $|\sigma| < 1$, we have that

$$\kappa e^{\sigma^2|\gamma\sigma\Delta t|} > e^{3\lambda^2|\gamma\sigma\Delta t|}\Big[\big|\widehat{u(t_1)}(\lambda + i\sigma)\big| - A\int_0^{|\Delta t|} e^{-3\lambda^2|\gamma\sigma|\tau}\,\Big|\widehat{F(t_1+\tau)}(\lambda + i\sigma)\Big|\,d\tau\Big],$$

with $A = e^{2|\gamma|T}$. Then,

$$\begin{aligned}\widehat{F(t)}(\lambda) = -\Big(&\lambda r_1(t)\Big[\widehat{u(t)} * \cdots * \widehat{u(t)}\Big(\lambda) \\ &+ \lambda r_2(t)\widehat{u(t)}(\lambda) + r_3(t)\widehat{u(t)}(\lambda)\Big).\end{aligned}$$

For $|\lambda| > 1$ large enough,

$$\begin{aligned}\big|\widehat{F(t)}(\lambda + i\sigma)\big| &< 2|\lambda|\Big[|r_1(t)|\big(|\widehat{u(t)}| * \cdots * |\widehat{u(t)}|\big)(\lambda + i\sigma) \\ &\quad + |r_2(t)||\widehat{u(t)}(\lambda + i\sigma)| + |r_3(t)||\widehat{u(t)}(\lambda + i\sigma)|\Big] \\ &< 2|\lambda|\Big[R_1\big(|\widehat{u(t)}| * \cdots * |\widehat{u(t)}|\big)(\lambda + i\sigma) \\ &\quad + (R_2 + R_3)|\widehat{u(t)}(\lambda + i\sigma)|\Big] \le 2|\lambda|\widehat{G(t)}(\lambda + i\sigma)\end{aligned}$$

where $R_i = \sup_{t\in I}|r_i(t)|$, and
$\widehat{G(t)}(z) = R_1\big(|\widehat{u(t)}| * \cdots * |\widehat{u(t)}|\big)(z) + (R_2 + R_3)|\widehat{u(t)}(z)|$.

Moreover, we also have that

$$\begin{aligned}&\big|\widehat{u(t_1)}(\lambda + i\sigma)\big| - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\gamma\lambda^2|\sigma\tau|}\big|\widehat{G(t_1+\tau)}(\lambda + i\sigma)\big|d\tau \\ &\ge \big|\widehat{u(t_1)}(\lambda)\big| - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\gamma|\sigma|\lambda^2\tau}\big|\widehat{G(t_1+\tau)}(\lambda)\big|d\tau \\ &\quad - \big|\widehat{u(t_1)}(\lambda + i\sigma) - \widehat{u(t_1)}(\lambda)\big| \\ &\quad - 2A|\lambda|\int_0^{|\Delta t|} e^{-3\gamma|\sigma|\lambda^2\tau}\big|\widehat{G(t_1+\tau)}(\lambda + i\sigma) - \widehat{G(t_1+\tau)}(\lambda)\big|d\tau,\end{aligned}$$

then using Lemma 2.4 and proceeding as in the proof of Theorem 3.1 we obtain the result. □

As a consequence of the previous Theorem, we obtain a uniqueness result for the (KdV) equation.

Corollary 3.1. *Let u, v solutions of the (KdV) equation*

$$u_t + \partial_x^3 u + u\partial_x u = 0 \quad en \;\; \mathbb{R}\times\mathbb{R}, \tag{48}$$

such that for $\delta > 0$, $u(t), v(t) \in H^{2+\delta}(\mathbb{R})$ and $\partial_x u(t), \partial_x v(t) \in \mathcal{A}$ uniformly for $t \in \mathbb{R}$. If $u(x,t) = v(x,t)$ for $(x,t) \in \Big([-B,B]\times I\Big)^c$, then $u(x,t) = v(x,t)$, for $(x,t) \in \mathbb{R}\times\mathbb{R}$.

Proof. Clearly, the Hölder inequality implies that $u(t), \partial_x u(t), \partial_x v(t) \in L^1(\mathbb{R})$ uniformly for $t \in I$. Thus we have that $w = u - v$ is a solution of the (KdV) type equation with variable coefficients

$$w_t + \partial_x^3 w + u(x,t)\partial_x w + \partial_x v(x,t) w = 0,$$

such that

$$\operatorname{supp} w(t)(\cdot) \subseteq [-B, B], \quad \text{for } t \in I.$$

Then from Theorem 3.1, we conclude that $u(x,y) = v(x,y)$ for $(x,t) \in [-B,B]\times I$. □

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Bibliography

[1] **Bourgain, J.** *On the compactness of the support of solutions of dispersive equations.* IMRN int. Math. Res. Notices **9** (1997), 437–447.

[2] **Grimshaw, R.** *Evolution equations for long nonlinear internal waves in stratified shear flows.* Stud. Appl. Math. **65** (1981), 159–188.

[3] **Johnson, R**. *On an asymptotic solution of the Korteweg–de Vries equation with slowly varying coefficients.* J. Fluid Mech., **60** (1973), 813–824.

[4] **Kenig, C.; Ponce, G.; Vega, L.** *On the Support of Solutions to the Generalized KdV Equation.* Ann. I. H. Poincaré **19** (2) (2002), 191–208.

[5] **Kenig, C.; Ponce, G.; Vega L.** *Higher–order nonlinear dispersive equations.*Proc. Amer. Math. Soc. **122** (1994), 157–166.

[6] **Saut, J.; Scheurer, B.** *Unique Continuation for Some Evolution Equations.* Journal Of Differential Equations **66** (1987),118–139.

[7] **Zhang, B.** *Unique Continuation for the Korteweg–de Vries equation.* Siam J. Math. Anal. **23** (1) (1992), 55–71.

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