A CHARACTERIZATION OF WEAKLY REGULAR LINEAR FUNCTIONALS

by

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Abstract

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A linear functional is said to be weakly-regular if it is not a finite sum of Dirac masses and their derivatives. In this paper, we consider the first-order linear differential equations $(Eu)' + Fu = 0$ where u is a non-zero linear functional and (E,F) is a pair of polynomials, with E monic. The aim of this work is to give weak-regularity conditions on u. Under certain admissibility conditions of the pair (E, F), the weak-regularity of u leads to its regularity. Some examples are analyzed.

Key words: First-order linear differential equations, weak-regular and regular functionals, weak-semiclassical and semi-classical functionals.

Resumen

Un funcional lineal se dice débilmente regular si no es la suma finita de masas de Dirac y sus derivadas. En este trabajo consideramos las ecuaciones diferenciales lineales de primer orden $(Eu)' + Fu = 0$, donde u es un funcional lineal no nulo y (E, F) es una pareja de polinomios, con E mónico. El propósito de este trabajo es dar condiciones de regularidad débil sobre u. Bajo ciertas condiciones de admisibilidad de la pareja (E, F), la regularidad débil de u conduce a su regularidad. Se analizan algunos ejemplos.

Palabras clave: Ecuaciones diferenciales lineales de primer orden, funcionales regulares y débilmente regulares, funcionales semiclásicos, funcionales débilmente semiclásicos.

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Introduction

Let u be a non-zero linear functional satisfying the following first-order linear differential equation

$$(Eu)' + Fu = 0 \qquad (*)$$

where E and F are non-zero polynomials, with E a monic polynomial.

When the linear functional u is regular then it is said to be semiclassical [6, 7].

Notice that a linear functional u is said to be regular [4, 7] if there exists a monic polynomial sequence (MPS) $\{B_n\}_{n\geq 0}$ where $\deg B_n = n$, $n \geq 0$, such that

$$\langle u, B_n B_m\rangle = r_n \delta_{n,m},\ n,\ m \geq 0, \quad r_n \neq 0,\ n \geq 0.$$

Besides regular functionals, the equation $(*)$ can have as solutions linear functionals defined as a finite sum of Dirac masses and their derivatives. In such a case there exists a non-zero polynomial ϕ such that $\phi u = 0$. More precisely, if $u = \sum_{i=1}^{t} \sum_{j=0}^{k_i-1} M_{i,j}\delta^{(j)}(x-x_i)$, then $\phi(x) = \prod_{i=1}^{t}(x-x_i)^{k_i}$. Obviously, such linear functionals u are not regular. For this reason, we introduce in a natural way the concept of weak-regularity linear functional u as follows.

A non-zero linear functional u is said to be *weakly-regular* if for a polynomial ϕ such that $\phi u = 0$, then $\phi = 0$. Regular linear functionals are weakly-regular (in general the converse is not true, see Remark 1.6).

In this paper, we are dealing with *weak-semiclassical linear functionals*, i.e., when the linear functional u satisfying $(*)$ is weakly-regular. The aim of our contribution is to give essentially a necessary and sufficient condition for the weak-regularity of a non-zero linear functional u satisfying $(*)$.

The paper is organized as follows. In Section 1, we introduce the basic notations and tools that will be used throughout the paper. Next, we define the weak-regularity of a linear functional and we analyze some properties like the stability by the shifting perturbation of the linear functional as well as the left multiplication of the linear functional by a polynomial. We conclude this section introducing the notion of admissible pair of polynomials. In section 2, our main results are proved. We obtain a necessary and sufficient condition in order to a non-zero linear functional u satisfying a first-order linear differential equation $(Eu)' + Fu = 0$ be weakly-regular. This yields the definition of weak-semiclassical functional. In section 3, we prove (Proposition 3.2) that the classical functionals are the only weakly-regular functionals satisfying $(Eu)' + Fu = 0$, where E and F are two polynomials, E monic, $\deg E \leq 2$, $\deg F = 1$, and the pair (E, F) is admissible. This result generalizes one by **Geronimus** on classical functionals, see [5]. In section 4, the results of section 3 are used to characterize semiclassical polynomial sequences, which are orthogonal with respect to regular functionals u given by $Au = \lambda Bv$, where A and B are two monic polynomials, $\lambda \in \mathbb{C}^*$, and v is a classical linear functional.

1. Definitions and background

Let $\mathbb{P}$ be the linear space of complex polynomials in one variable and $\mathbb{P}'$ its topological dual space. We denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_{c,n} := \langle u, (x-c)^n\rangle$, $n \geq 0$, the moments of u with respect to the sequence $\{(x-c)^n\}_{n\geq 0}$. In particular, if $c = 0$, then we will denote $(u)_n := (u)_{0,n}$, $n \geq 0$.

We define the following operations in $\mathbb{P}'$. For any linear functional u, any polynomial h, and any $c \in \mathbb{C}$, let $Du = u'$, hu, $(x-c)^{-1}u$, and $\sigma(u)$ be the linear functionals defined by duality

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad f \in \mathbb{P},$$
$$\langle hu, f\rangle := \langle u, hf\rangle,\ f \in \mathbb{P},$$
$$\langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c(f)\rangle,\ f \in \mathbb{P},$$
$$\langle \sigma(u), f\rangle := \langle u, \sigma(f)\rangle,\ f \in \mathbb{P},$$

where $\theta_c(f)(x) = \dfrac{f(x)-(c)}{x-c}$ and $\sigma(f)(x) = f(x^2)$. Notice that

$$f(x)\sigma(u) = \sigma\big(f(x^2)u\big),\ f \in \mathbb{P}. \qquad (1.1)$$

Let $\{B_n\}_{n\geq 0}$ be a monic polynomial sequence (MPS), $\deg B_n = n$, $n \geq 0$, and $\{u_n\}_{n\geq 0}$ its dual sequence, $u_n \in \mathbb{P}'$, $n \geq 0$, defined by $\langle u_n, B_m\rangle := \delta_{n,m}$, $n,\ m \geq 0$, where $\delta_{n,m}$ is the Kronecker symbol.

The linear functional u_0 is said to be the canonical functional associated with the MPS $\{B_n\}_{n\geq 0}$.

We remind the following results [2, 4, 7].

Lemma 1.1. *For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent*

i) $\langle u, B_{m-1}\rangle \neq 0, \quad \langle u, B_n\rangle = 0, n \geq m.$
ii) *There exist $\lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m-1, \lambda_{m-1} \neq 0$ such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.*

ii) $\Rightarrow$ *i)*. Let ϕ be a polynomial such that $\phi u = 0$. We can always write $\phi(x) = A(x^2) + xB(x^2)$ where A and B are polynomials. For every $p \in \mathbb{P}$, one has

$$\begin{aligned} 0 &= <\phi u, p(x^2)> = <u, \phi(x)p(x^2)> \\ &= <u, A(x^2)p(x^2)> = <\sigma(u), A(x)p(x)> \\ &= <A\sigma(u), p(x)>. \end{aligned}$$

Then, $A\sigma(u) = 0$. So, from the weak-regularity of $\sigma(u)$ we deduce $A = 0$. On the other hand,

$$\begin{aligned} 0 &= <\phi u, xp(x^2)> = <u, x\phi(x)p(x^2)> \\ &= <u, x^2B(x^2)p(x^2)> = <\sigma(u), xB(x)p(x)> \\ &= <xB\sigma(u), p(x)>. \end{aligned}$$

Then, $xB\sigma(u) = 0$. So, $B = 0$ taking into account the weak-regularity of $\sigma(u)$. Thus $\phi = 0$. □

Notice that if u is a symmetric regular linear functional then xu is a weakly-regular functional that is not regular.

1.2. **Admissibility pair of polynomials.** Let (E, F) be a pair of polynomials, where E monic, $\deg E = t$, $\deg F = p \geq 1$, and $s(E,F) := \max(t-2,\ p-1)$. Notice that ($s(E,F) \geq 0$, because $\deg F \geq 1$). For this pair of polynomials, we introduce

- the admissibility coefficients

$$\Delta_n(E,F) = nE^{(s+2)}(0) - (s+2)F^{(s+1)}(0),\ n \geq 0; \quad (1.6)$$

•• the sequence of polynomials

$$F_m = F - (m-1)E', \quad m \geq 1. \quad (1.7)$$

Definition 1.8. The pair (E, F) is said to be admissible when its admissibility coefficients satisfy

$$\Delta_n(E,F) \neq 0,\ n \geq 0. \quad (1.8)$$

From an admissible pair of polynomials, we can deduce other admissible pairs. Indeed, we have the following result.

Lemma 1.9. *When (E, F) is admissible, then for each integer $m \geq 1$, we have*

i) $\deg F_m \geq 1$, *where* $F_m = F - (m-1)E'$, $m \geq 1$.
ii) $s(E, F_m) = s(E, F) := s$.
iii) *The pair (E, F_m) is admissible and*

$$\Delta_n(E, F_m) = \Delta_{n+(m-1)(s+2)}(E,F), \quad n \geq 0.$$

Proof. Assume there exists an integer, $m \geq 1$, such that F_m is a constant polynomial. Since $\deg F \geq 1$, then $m \geq 2$. In this case, $s = t - 2 = p - 1$ and the coefficient of x^p in F_m is $(p!)^{-1}F^{(p)}(0) - (m-1)t = 0$. Then $\Delta_{(m-1)t}(E,F) = 0$, and this contradicts the admissibility condition of the pair (E,F). Hence, i) holds. The admissibility condition of the pair (E, F) yields

$$deg(F_m) = max(p, t-1) = s+1, \quad m \geq 1. \quad (1.9)$$

Thus,

$$s(E, F_m) = max(t-2, s) = s.$$

Hence, ii) holds.

From i), ii), and (1.6), one has

$$\begin{aligned} \Delta_n(E,F_m) &= nE^{(s+2)}(0) - (s+2)F_m^{(s+1)}(0) \\ &= \big(n + (m-1)(s+2)\big)E^{(s+2)}(0) \\ &\qquad - (s+2)F^{(s+1)}(0) \\ &= \Delta_{n+(m-1)(s+2)}(E,F), \quad n \geq 0. \end{aligned}$$

Thus, the admissibility condition of the pair (E, F_m) follows from the admissibility of the pair (E,F). Hence, iii) holds. □

For each fixed $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, we can consider the shifted pair $(\tilde{E}, \tilde{F})$ given by

$$\tilde{E}(x) := a^{-t}E(ax+b) \quad ; \quad \tilde{F}(x) := a^{1-t}F(ax+b). \quad (1.10)$$

Let denote $\tilde{s} = \max(\tilde{t} - 2,\ \tilde{p} - 1)$, where $\tilde{t} = \deg(\tilde{E})$ and $\tilde{p} = \deg(\tilde{F})$. Thus

$$\tilde{t} = t, \quad \tilde{p} = p, \quad \tilde{s} = s. \quad (1.11)$$

As a consequence the following result holds.

Lemma 1.10. *If (E, F) is admissible, then $(\tilde{E}, \tilde{F})$ is also admissible. Furthermore,*

$$\Delta_n(\tilde{E},\tilde{F}) = a^{s+2-t}\Delta_n(E,F),\ n \geq 0, \quad (1.12)$$

Proof. If $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, then

$$\begin{aligned} \Delta_n(\tilde{E},\tilde{F}) &= n\tilde{E}^{(s+2)}(0) - (s+2)\tilde{F}^{(s+1)}(0) \\ &= a^{s+2-t}\Big(nE^{(s+2)}(0) - (s+2)F^{(s+1)}(0)\Big) \\ &= a^{s+2-t}\Delta_n(E,F),\ n \geq 0. \end{aligned}$$

Hence, i) follows. □

When the pair of polynomials (E, F) is admissible and $\deg E \geq 1$, we deduce the following results.

Lemma 1.11. *Let (E,F) be an admissible pair of polynomials, where $\deg E \geq 1$, and c is a zero of E. Then for each integer $m \geq 1$, we get*

i) $\deg \hat{F}_m \geq 1$, *where* $\hat{F}_m = F - (m-1)\theta_c(E)$.
ii) $s(E,\hat{F}_m) = s(E,F) := s$.
iii) *The pair* $(E,\hat{F}_m)$ *is admissible, and*

$$\Delta_n(E,\hat{F}_m) = \Delta_{n+m-1}(E,F), \quad n \geq 0.$$

Proof. Assume there exists an integer $m \geq 1$ such that $\hat{F}_m$ is a constant polynomial. Since $\deg F \geq 1$, then $m \geq 2$. In this case, $s = t-2 = p-1$ and the coefficient of x^p in $\hat{F}_m$ is $(p!)^{-1}F^{(p)}(0) - (m-1) = 0$. Then $\Delta_{m-1}(E,F) = 0$, and this contradicts the admissibility condition of the pair (E,F). Hence, i) holds. The admissibility condition of the pair (E,F) means that

$$\deg(\hat{F}_m) = \max(p, t-1) = s+1, \quad m \geq 1. \tag{1.13}$$

Thus,

$$s(E,\hat{F}_m) = \max(t-2, s) = s, \quad m \geq 1.$$

Hence, ii) holds. For $m \geq 1$, from i), ii), and (1.6) one has

$$\Delta_n(E,\hat{F}_m) = nE^{(s+2)}(0) - (s+2)\hat{F}_m^{(s+1)}(0).$$

Since $t \leq s+2$, it follows that

$$E^{(s+2)} = (s+2)\big(\theta_c(E)\big)^{(s+1)}.$$

Thus,

$$\begin{aligned}\Delta_n(E,\hat{F}_m) &= (n+m-1)E^{(s+2)}(0) - (s+2)F^{(s+1)}(0)\\ &= \Delta_{n+m-1}(E,F), \quad n \geq 0.\end{aligned}$$

Hence, iii) holds. □

For the sequel, we need the following results.

Lemma 1.12. *Let (E,F) be a pair of non-zero polynomials, where E monic, $\deg E \geq 1$. If E and F are coprime, then*

i) *There exists an integer $\mu \geq 1$ such that E and $F_m = F - (m-1)E'$ are coprime, $m \geq \mu$.*
ii) *For each zero c of E, there exists an integer $\vartheta \geq 1$ such that E and $\hat{F}_m = F - (m-1)\theta_c(E)$ are coprime, $m \geq \vartheta$.*

Proof. Assume that for each integer $\mu \geq 1$, there exists an integer $m_\mu \geq \mu$ such that E and $F - (m_\mu - 1)E'$ have a common zero. Then there is a zero c of E and two different integers $m_{\mu_\nu} \geq 1$, $\nu = 1,2$ respectively, such that $\big(F - (m_{\mu_\nu} - 1)E'\big)(c) = 0$, $\nu = 1,2$. This yields $F(c) = 0$, that contradicts the fact that E and F are coprime. Hence, i) holds. Let c be a zero of E. Two cases must be analyzed.

Case1. Let assume c is a simple zero of E. Suppose that for each integer $\vartheta \geq 1$, there exists an integer $m_\vartheta \geq \vartheta$ such that E and $F - (m_\vartheta - 1)\theta_c(E)$ have a common zero. Then it will exist a zero c of E and two different integers $m_{\vartheta_\nu} \geq 1$, $\nu = 1,2$ such that $\big(F - (m_{\vartheta_\nu} - 1)\theta_c(E)\big)(c) = 0$, $\nu = 1,2$. This leads to $F(c) = 0$, in contradiction with the fact that E and F are coprime.

Case2. c is a zero of E with multiplicity at least two. For every zero ξ of E, we have

$$\hat{F}_m(\xi) = F(\xi) - (m-1)\theta_c(E)(\xi) = F(\xi) \neq 0, \ m \geq 1.$$

Hence, ii) holds. □

As a consequence, for a pair of non-zero polynomials (E,F), where E is a monic polynomial, $\deg E \geq 1$, and where E and F are coprime, we can associate the integer

$$\mu(E,F) := \min\{k \geq 1 : E \text{ and } F_m \text{ are coprime}, m \geq k\}. \tag{1.14}$$

2. Main Results

Let (E,F) be a pair of polynomials, with E monic, $\deg E = t$, $\deg F = p \in \mathbb{N} \cup \{-\infty\}$, and $s := s(E,F)$. Consider the functional equation

$$(Eu)' + Fu = 0, \quad u \in \mathbb{P}'^{*}. \tag{2.1}$$

Lemma 2.1. *Let $u \in \mathbb{P}'^{*}$ a solution of (2.1). When the pair (E,F) is admissible and the $(s+1)-$ first moments $(u)_0$, ..., $(u)_s$ are fixed, then u is unique.*

Proof. The admissibility condition of the pair (E,F) requires that $p \geq 1$. Then, $s \geq 0$. The functional equation (2.1) is equivalent to the following recurrence relation for the corresponding moments

$$\sum_{\nu=0}^{s+2} \frac{\Delta_{n,\nu}(E,F)}{\nu!}(u)_{n+\nu-1} = 0, \ n \geq 0, \tag{2.2}$$

where $\Delta_{n,\nu}(E,F) := nE^{(\nu)}(0) - \nu F^{(\nu-1)}(0)$, $0 \leq \nu \leq s+2$. Suppose that $v \in \mathbb{P}'^{*}$ is other solution of (2.1). Then, the linear functional $w = v - u$ satisfies

$$\sum_{\nu=0}^{s+2} \frac{\Delta_{n,\nu}(E,F)}{\nu!}(w)_{n+\nu-1} = 0, \ n \geq 0,$$

where $F_\nu = F-(\nu-1)E'$. Notice that $A'E - AF_\nu \neq 0$,. Otherwise, $A'E = AF_\nu$. Since E and F_ν are coprime E divides A, a contradiction. Taking into account ϕ is a polynomial of minimal degree such that $\phi u = 0$ then $\phi = AE^\nu$ divides $E^{\nu-1}(A'E - AF_\nu)$. So, E divides AF_ν. But E and F_ν are coprime then E divides A, a contradiction. Thus,

$$\phi(x) = E^k(x).$$

From (2.5) and from (2.1) , we get

$$E^{k-1}F_k u = 0.$$

Since E and F_k are coprime, there exist two polynomials S_i, $i = 1, 2$ such that

$$S_1(x)E(x) + S_2(x)F_k(x) = 1.$$

Then,

$$S_1(x)E^k(x) + S_2(x)E^{k-1}(x)F_k(x) = E^{k-1}(x).$$

Multiplying by u, we get $E^{k-1}u = 0$. This contradicts the fact that $\phi = E^k$ has minimal degree and satisfies $\phi u = 0$. Hence, the weak regularity of u follows. □

$\mathbf{A}_{2.2}$. E and F are coprime and $\mu(E,F) \geq 2$.

Lemma 2.8. *Let $u \in \mathbb{P}'^*$ satisfy (2.1), with pseudo-class $t \geq 1$, E and F coprime polynomials, and $\mu(E,F) \geq 2$. Then the following statements are equivalent.*

i) *u is weakly-regular.*
ii) $E^{\mu(E,F)-1}u \neq 0$.

Proof. From the assumption, let consider the linear functional $v = E^{\mu(E,F)-1}u$. From Property 1.5, i), u is a weakly-regular linear functional if and only if $v \neq 0$ and v is weakly-regular. From (2.1), when $v \neq 0$, it satisfies

$$(Ev)' + F_{\mu(E,F)}v = 0,$$

where E and $F_{\mu(E,F)} - (m-1)E' = F_{m+\mu(E,F)-1}$ are coprime, $m \geq 1$. Since E and $E' + F_{\mu(E,F)} = F_{\mu(E,F)-1}$ are coprime, then the pseudo-class of v is $t \geq 1$. Therefore, from Lemma 2.7 v is weakly-regular. Hence, u is weakly-regular if and only if $E^{\mu(E,F)-1}u \neq 0$. □

$\mathbf{A}_{2.3}$. E and F are not coprime. Let denote Δ the greatest common divisor of E and F, with $E = \Delta\tilde{E}$ and $F = \Delta\tilde{F}$, $\deg\Delta \geq 1$. Moreover, we can associate with the pair of polynomials $(\tilde{E},\tilde{F})$ the integer $\mu(\tilde{E},\tilde{F})$.

Proposition 2.9. *Let $u \in \mathbb{P}'^*$ be a linear functional such that (2.1) holds with pseudo-class $t \geq 1$, and G be the greatest common divisor of E and F, with $E = G\tilde{E}$ and $F = G\tilde{F}$. The following statements are equivalent.*

i) *u is weakly-regular.*
ii) (i). *If $\deg\tilde{E} = 0$, then $\deg\tilde{F} \geq 1$ and $Gu \neq 0$.*
(ii). *If $\deg\tilde{E} \geq 1$, then $G\tilde{E}^{\mu(\tilde{E},\tilde{F})-1}u \neq 0$.*

Proof. Consider $v = G\tilde{E}^{\mu(\tilde{E},\tilde{F})-1}u$. The linear functional u is weakly-regular if and only if $v \neq 0$ and v is weakly-regular. But, if $v \neq 0$, then

$$(\tilde{E}v)' + \tilde{F}_{\mu(\tilde{E},\tilde{F})}v = 0,$$

where $\tilde{E}$ and $\tilde{F}_{\mu(\tilde{E},\tilde{F})} - (m-1)\tilde{E}' = \tilde{F}_{m+\mu(\tilde{E},\tilde{F})-1}$ are coprime, $m \geq 1$. Since $\tilde{E}$ and $\tilde{E}' + \tilde{F}_{\mu(\tilde{E},\tilde{F})} = \tilde{F}_{\mu(\tilde{E},\tilde{F})-1}$ are coprime, then $\tilde{t} = \deg\tilde{E}$ is the pseudo-class of v. Two cases appear.

(i). $\tilde{t} = 0$. According to Lemma 2.6 the non-zero linear functional v is weakly-regular if and only if $\deg\tilde{F} \geq 1$. In this case, u is weakly-regular if and only if $Gu \neq 0$ and $\deg\tilde{F} \geq 1$.

(ii). $\tilde{t} \geq 1$. The non-zero linear functional v is weakly-regular, from Lemmas 2.7 and 2.8. In this case, u is weakly-regular if and only if $G\tilde{E}^{\mu(\tilde{E},\tilde{F})-1}u \neq 0$. □

Remark 2.10. When the linear functional u solution of (2.1) satisfies $(u)_0 \neq 0$, and is weakly-regular, then we must have $\deg F \geq 1$. If not, $F(x) = \lambda \in \mathbb{C}$, then $(Eu)' + \lambda u = 0$ holds. So, from $\langle (Eu)' + \lambda u, 1\rangle = 0$, we get $\lambda(u)_0 = 0$. Hence, $\lambda = 0$, and, as a consequence, $Eu = 0$. This contradicts the weak-regularity of u.

2.2. Weak-semiclassical and semiclassical functionals.

Let introduce the following definitions.

Definition 2.11. The linear functional u is said to be a weak-semiclassical functional when it is weakly-regular and satisfies (2.1), where the pair (E,F) is admissible.

Notice that every semiclassical linear functional u is also regular [7]. A weak-semiclassical functional u satisfies an infinity number of first-order linear differential equations: for $\chi \in \mathbb{P}$, u also fulfils

$$(E_1u)' + F_1u = 0,$$

with $E_1(x) = \chi(x)E(x)$, and $F_1(x) = \chi(x)F(x) - \chi'(x)E(x)$. So, if $s = s(E,F) = \max(t-2, p-1)$ and taking into account the admissibility condition of the pair (E,F), i.e $\Delta_q(E,F) \neq 0$, then we get $s_1 =$

$s(E_1, F_1) = s + q$. Hence, we can associate with the weak-semiclassical functional u a subset $h(u)$ of nonnegative integers such that m belongs to $h(u)$ if and only if $m = s(E_2, F_2)$ where (E_2, F_2) is an admissible pair of polynomials satisfying (2.1).

Definition 2.12. The minimum element s of $h(u)$ is said to be the class of u. When $s = 0$, the weak-semiclassical (resp. semiclassical) functional is called weak-classical (resp. classical) functional.

Lemma 2.13. *Let u be a weak-semiclassical functional such that*

$$(E_i u)' + F_i u = 0, \quad with \ s_i = \max(t_i - 2, p_i - 1), \quad i = 1, 2.$$

Let denote by E the greatest common divisor of E_1 and E_2. Then, there exists a polynomial F such that

$$(Eu)' + Fu = 0,$$

with $s = \max(t-2, p-1) = s_i - t_i + t$, $i = 1, 2$, where $t = \deg E$ and $p = \deg F$.

Proof. See in [8] Lemma 3.3 and replace regularity by weak-regularity. □

Proposition 2.14. *For each weak-semiclassical functional u, the pair (E, F) that realizes the minimum of $h(u)$ is unique.*

Proof. See in [8] Proposition 3.4 and replace regularity by weak-regularity. □

Proposition 2.15. *The class of the weak-semiclassical functional u satisfying (2.1) is s if and only if*

$$\prod_c \left(| F(c) + E'(c) | + | \langle u, \theta_c F + \theta_c^2 E \rangle | \right) > 0,$$

where c belongs to the set of zeros of E.

Proof. See in [8] Proposition 3.5 and replace regularity by weak-regularity. □

Proposition 2.16. *Let u be a weak-semiclassical functional satisfying $(Eu)' + Fu = 0$, where E monic, $t = \deg E$, $p = \deg F \geq 1$, and $s = \max(t-2, p-1)$. The following statements are equivalent.*

i) *The pseudo-class of u is t.*
ii) *The class of u is s.*

Proof. It is a straightforward consequence of Lemmas 2.3 and 2.13. □

Remark 2.17. Let u be a weak-semiclassical functional satisfying (2.1), with $\deg E \geq 1$. For each zero c of E and an integer $m \geq 1$, let consider the following linear functional

$$v(m, c) = (x - c)^{m-1} u.$$

Obviously, $v(m, c)$ is weakly-regular and satisfies

$$(Ev(m, c))' + \hat{F}_m v(m, c) = 0.$$

From Lemma 2.9, the pair of polynomials $(E, \hat{F})$ is admissible and has associated a nonnegative integer number s. Thus, there exists an integer number $k \geq 1$ such that $(v(k, c))_0 \neq 0$. Otherwise, one has $\langle u, (x-c)^{m-1} \rangle = 0$, $m \geq 1$. Then $u = 0$, a contradiction.

3. Classical Case.

It is well known that if $s = 0$ and the linear functional u is regular then we recover the classical functionals (Hermite, Laguerre, Bessel, and Jacobi) [1, 9, 10]. By a shift we get the following canonical classical functionals

$\mathbf{C}_1$. $E(x) = 1$, $F(x) = 2x$.

The functional u is the Hermite functional denoted $\mathcal{H}$.

$\mathbf{C}_2$. $E(x) = x$, $F(x) = x - \alpha - 1$.

The functional u is the Laguerre functional denoted $\mathcal{L}(\alpha)$. It is regular if and only if $\alpha \neq -n$, $n \geq 1$.

$\mathbf{C}_3$. $E(x) = x^2$, $F(x) = -2(\alpha x + 1)$.

The functional u is the Bessel functional denoted $\mathcal{B}(\alpha)$. It is regular if and only if $\alpha \neq -\frac{n}{2}$, $n \geq 0$.

$\mathbf{C}_4$. $E(x) = x^2 - 1$, $F(x) = -(\alpha + \beta + 2)x + \alpha - \beta$.

The functional u is the Jacobi functional denoted $\mathcal{J}(\alpha, \beta)$. It is regular if and only if $\alpha \neq -n$, $\beta \neq -n$, and $\alpha + \beta \neq -n - 1$, $n \geq 1$.

Notice that the polynomials (E, F) in the above four canonical classical cases, $\mathbf{C}_i$, $i = 1, ..., 4$, are coprime, $m \geq 1$.

In the theory of first-order linear differential equations, the weak-regularity of the functional could reach its regularity, what is true here. First, we need to show the invariance of the weak-semiclassical character by shifting.

Lemma 3.1. *When u is a weak-semiclassical functional of class s, satisfying (2.1), then $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ is also a weak-semiclassical functional with the same class s. It satisfies $(\tilde{E}\tilde{u})' + \tilde{F}\tilde{u} = 0$, where $\tilde{E}(x) = a^{-t}E(ax+b)$ and $\tilde{F}(x) = a^{1-t}F(ax+b)$.*

Proof. The weak-regularity of $\tilde{u}$ and the admissibility conditions of the pair $(\tilde{E}, \tilde{F})$ follow from Property 1.5, b) and Lemma 1.10, respectively. Finally, for the functional equation and the class of $\tilde{u}$, see [1]. □

Proposition 3.2. *Let u be a linear functional satisfying $(Eu)' + Fu = 0$, where E monic, $\deg E \le 2$, $\deg F = 1$, and the pair of polynomials (E, F) is admissible. The following statements are equivalent.*

i) *u is regular.*
ii) *For each integer $m \ge 1$, E and F_m are coprime.*
iii) *u is weakly-regular.*

Proof. i) ⇒ ii)⇒ iii). It is straightforward.

iii) ⇒ i). It is sufficient to show that u is regular. So the following four situations must be analyzed:

C$_1$. $\deg(E) = 0$. We can write $E(x) = 1$ and $F(x) = cx + d$, $c \neq 0$. The shifted functional $v = (h_{a^{-1}} \circ \tau_{-b})u$, where $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ such that $a^2 = (2/c)$ and $b = -(d/c)$, satisfies

$$v' + 2xv = 0, \quad (v)_0 = 1. \tag{3.1}$$

The Hermite functional is the unique solution of (3.1). Hence, u is regular as the shifted of a regular functional.

C$_2$. $\deg(E) = 1$. We can write $E(x) = x + \xi$ and $F(x) = cx + d$, $c \neq 0$. Let $v = (h_{a^{-1}} \circ \tau_{-b})u$, where $a = (1/c)$, $b = -\xi$ and $\alpha = c\xi - d - 1$. The functional v satisfies

$$(xv)' + (x - \alpha - 1)v = 0, \quad (v)_0 = 1. \tag{3.2}$$

Applying (3.2) to x^n, $n \ge 0$, we get

$$(v)_{n+1} = [n - (\alpha+1)](v)_n, \; n \ge 0, \quad (v)_0 = 1. \tag{3.3}$$

Notice that $\alpha \neq -n$, $n \ge 1$. Otherwise, there exists an integer n_0, $n_0 \ge 0$, such that $\alpha = -n_0 - 1$. From (3.3), one has $x^{n_0+1}v = 0$. This contradicts the weak-regularity of v, as the shifted of a weakly-regular functional. Therefore, v is the Laguerre functional. Thus, u is regular.

C$_3$. $\deg(E) = 2$ and E has a double zero. We can write $E(x) = (x+\xi)^2$ and $F(x) = cx + d$. Since $\deg(F) = 1$ and taking into account (E, F) is an admissible pair we get $c \neq n$, $n \ge 0$. If $\alpha = -(c/2)$, then $\alpha \neq -(n/2)$, $n \ge 0$. Let $a = \dfrac{c\xi - d}{2}$. Then, $a \neq 0$. Otherwise, the functional $v = \tau_\xi u$ satisfies

$$(x^2 v)' - 2\alpha x v = 0, \tag{3.4}$$

and applying (3.4) to x^n, $n \ge 0$, we get $(n+2\alpha)(v)_{n+1} = 0$, $n \ge 0$. Since $\alpha \neq -(n/2)$, $n \ge 0$, then $xv = 0$, and this leads to a contradiction. So, it is possible to consider the functional $v = (h_{a^{-1}} \circ \tau_{-b})u$, where $a = \dfrac{c\xi - d}{2}$ and $b = -\xi$. The shifted functional v satisfies

$$(x^2 v)' - 2(\alpha x + 1)v = 0, \; (v)_0 = 1, \tag{3.5}$$

where $\alpha \neq -(n/2)$, $n \ge 0$. Thus, v is the Bessel functional and u is regular.

C$_4$. $\deg(E) = 2$ and E has two different zeros. We can write $E(x) = (x+\xi_1)(x+\xi_2)$, with $\xi_1 \neq \xi_2$, and $F(x) = cx + d$, where $c \neq n$, $n \ge 0$. Let $v = (h_{a^{-1}} \circ \tau_{-b})u$, where $a = \dfrac{\xi_2 - \xi_1}{2}$ and $b = \dfrac{\xi_1 + \xi_2}{2}$. We take $\alpha = \dfrac{c(b-a) + d - 2a}{2a}$ and $\beta = -\dfrac{c(a+b) + d + 2a}{2a}$. The shifted functional v satisfies

$$((x^2-1)v)' + (-(\alpha+\beta+2)x + \alpha - \beta)v = 0, \; (v)_0 = 1, \tag{3.6}$$

with $\alpha + \beta = -c - 2 \neq -n - 2$, $n \ge 0$. Applying (3.6) to $(x-1)^n$, $n \ge 0$,

$$(n+\alpha+\beta+2)v_{1,n+1} = -2(n+\beta+1)v_{1,n}, \; n \ge 0, \tag{3.7}$$

On the other hand, applying (3.6) to $(x+1)^n$, $n \ge 0$, we get

$$(n+\alpha+\beta+2)v_{-1,n+1} = 2(n+\alpha+1)v_{-1,n}, \; n \ge 0. \tag{3.8}$$

Suppose there exists an integer n_0, $n_0 \ge 0$, such that $\beta = -n_0 - 1$ (resp. $\alpha = -n_0 - 1$). Since $\alpha + \beta \neq -n - 2$, $n \ge 0$, from (3.7), (resp. (3.8)), then $(x-1)^{n_0+1}v = 0$, (resp. $(x+1)^{n_0+1}v = 0$). This contradicts the weak-regularity of v.

As a consequence, $\alpha + \beta \neq -n$, $n \ge 2$, $\alpha \neq -n$, $n \ge 1$, and $\beta \neq -n$, $n \ge 1$. The functional v is the Jacobi functional, then u is regular. □

Proposition 3.3. *Let $\{C_n\}_{n\ge0}$ be a sequence of monic polynomials with dual sequence $\{c_n\}_{n\ge0}$, such that $E(x)C''_{n+1}(x) - F(x)C'_{n+1}(x) = \lambda_{n+1}C_{n+1}(x)$, $n \ge 0$, where E monic, $\deg E \le 2$, $\deg F = 1$, and the pair of polynomials (E, F) is admissible. The following statements are equivalent.*

i) *$\{C_n\}_{n\ge0}$ is orthogonal with respect to c_0.*

ii) *For each integer* $m \geq 1$, E *and* F_m *are coprime.*
iii) c_0 *is weakly-regular.*

Proof. From the higher degree coefficients in the second-order differential equation, and the admissibility condition of the pair (E, F), we get

$$\lambda_{n+1} = (n+1)\big(n\frac{E''(0)}{2} - F'(0)\big) \neq 0, \; n \geq 0. \qquad (3.9)$$

On the other hand, we get

$$(Ec_0)' + Fc_0 = 0. \qquad (3.10)$$

i)⇒ ii). It is a consequence of (3.10), the regularity of c_0, and Proposition 3.2.

ii)⇒ iii). It follows from Proposition 3.2.

iii)⇒ i). From Proposition 3.2, the linear functional c_0 is regular. Thus, the sequence $\{C_n\}_{n\geq 0}$ will be orthogonal with respect to c_0, see in [8] Proposition 2.9. □

4. Applications.

A$_1$. Assume that v is a classical functional. Let u be a regular functional such that

$$Au = \lambda Bv. \qquad (4.1)$$

Here $\lambda \in \mathbb{C}^*$ and A, B are two monic polynomials.This kind of perturbations have been analyzed in [11]. The linear functional u is semi-classical. Indeed, if we assume that the functional v satisfies $Ev' + Fv = 0$, where E monic, $\deg E \leq 2$, $\deg F = 1$, and the pair (E, F) is admissible, then it is easy to prove that u satisfies

$$(ABu)' + A(BF - 2B'E)u = 0. \qquad (4.2)$$

From Proposition 3.2, we can characterize in a natural way the MOPS with respect to u. Indeed

Proposition 4.1. *Let* B *be a monic polynomial,* $degB = t$, *and* $\{B_n\}_{n\geq 0}$ *be the sequence of monic orthogonal polynomials with respect to* u. *The following statements are equivalent.*

i) *There exist a monic polynomial* A, *a non zero-constant* λ, *and a classical functional* v *such that*

$$Au = \lambda Bv. \qquad (4.3)$$

ii) *There exist a integer* $s \geq 0$, *a MPS* $\{\Pi_{n+t}\}_{n\geq s}$, $\deg \Pi_{n+t} = n+t$, $n \geq s$, *and non zero-constants* ϑ_n, $n \geq s$, *such that*

$$\vartheta_n B(x)B_{n+1}(x) = E(x)\Pi'_{n+t}(x) - F(x)\Pi_{n+t}(x), \quad n \geq s, \qquad (4.4)$$

where (E, F) *is an admissible pair of polynomials,* E *monic,* $\deg E \leq 2$, $\deg F = 1$, *and* E *and* F_m *are coprime,* $m \geq 1$.

Proof. i)⇒ ii). Let $\{C_n\}_{n\geq 0}$ be the MOPS with respect to the functional v. From Lemma 2.1, if $s = degA$, then

$$B(x)B_{n+1}(x) = \sum_{\nu=n-s}^{n+t} \frac{\langle u, AC_{\nu+1}B_{n+1}\rangle}{\lambda\langle v, C_{\nu+1}^2\rangle} C_{\nu+1}(x), \; n \geq s. \qquad (4.5)$$

On the other hand the classical sequence $\{C_n\}_{n\geq 0}$ satisfies a second-order differential equation [3]

$$E(x)C''_{\nu+1}(x) - F(x)C'_{\nu+1}(x) = \lambda_{\nu+1}C_{\nu+1}(x), \; \nu \geq 0, \qquad (4.6)$$

where $\lambda_{\nu+1} = (\nu+1)\big(\nu\frac{E''(0)}{2} - F'(0)\big) \neq 0, \; \nu \geq 0.$ Using (4.6), from (4.5) we deduce (4.4), with

$$\vartheta_n = \frac{\lambda_{n+t+1}}{n+t+1}, \; n \geq s, \qquad (4.7)$$

$$\Pi_{n+t}(x) = \sum_{\nu=n-s}^{n+t} \frac{\lambda_{n+t+1}}{\lambda_{\nu+1}} \frac{\langle u, AC_{\nu+1}B_{n+1}\rangle}{\lambda\langle v, C_{\nu+1}^2\rangle(n+t+1)} C'_{\nu+1}(x), \qquad (4.8)$$

for $n \geq s$.

ii) ⇒ i). From the assumption *ii)* and Proposition 3.2, let consider the classical functional v satisfying $(Ev)' + Fv = 0$. From (4.4), we get $\langle Bv, B_{n+1}\rangle = 0, \; n \geq s$. Thus, there exists an integer r, $0 \leq r \leq s$, such that $\langle Bv, B_r\rangle \neq 0$. Otherwise, since $\langle Bv, B_n\rangle = 0, \; n \geq 0$, then $Bv = 0$. This contradicts the regularity of v. As a consequence, $\langle Bv, B_{n+1}\rangle = 0, \; n \geq s$, and $\langle Bv, B_r\rangle \neq 0$. From Lemma 2.1, we get $Bv = \sum_{\nu=0}^{r} \langle Bv, B_\nu\rangle \, u_\nu$, and by using (1.4), we finally obtain (4.3), with

$$\lambda = \frac{\langle u, B_r^2\rangle}{\langle Bv, B_r\rangle},$$

$$A(x) = \sum_{\nu=0}^{r} \frac{\langle Bv, B_\nu\rangle}{\langle Bv, B_r\rangle} \frac{\langle u, B_r^2\rangle}{\langle u, B_\nu^2\rangle} B_\nu(x). \qquad □$$

A$_2$. For each fixed $\mu \in \mathbb{C}^*$, let $u(\mu)$ be the linear functional satisfying

$$\big(Eu(\mu)\big)' + Fu(\mu) = 0, \; \big(u(\mu)\big)_0 = 1, \; \big(u(\mu)\big)_1 = 0, \qquad (4.9)$$

with $E(x) = x$ and $F(x) = 2x^2 - (2\mu + 1)$. If $\big(u(\mu)\big)_n$, $n \geq 0$, denote the moments of $u(\mu)$, we get

$$\big(u(\mu)\big)_{n+2} = \frac{(n+2\mu+1)}{2}\big(u(\mu)\big)_n, \; n \geq 0, \quad \big(u(\mu)\big)_1 = 0, \; \big(u(\mu)\big)_0 = 1. \qquad (4.10)$$

Clearly $u(\mu)$ is a symmetric linear functional.

Proposition 4.2. *For each fixed $\mu \in \mathbb{C}^*$, let $u(\mu)$ be the linear functional satisfying* (4.9). *The following statements are equivalent.*

i) *$u(\mu)$ is regular.*
ii) *$u(\mu)$ is weakly-regular.*
iii) *The linear functional $\sigma(u(\mu))$ is weakly-regular.*

Proof. i) $\Rightarrow$ ii). The regularity of $u(\mu)$ yields weak-regularity.

ii) $\Rightarrow$ *iii)*. According to Proposition 1.7 and taking into account that $u(\mu)$ is symmetric and weakly-regular, we deduce that $\sigma(u(\mu))$ is weakly-regular.

iii) $\Rightarrow$ *i)*. From (4.9) the linear functional $\sigma(u(\mu))$ satisfies

$$\big(x\sigma(u)\big)' + (x-\alpha-1)\sigma(u) = 0, \alpha = \mu - \frac{1}{2}. \quad (4.11)$$

From Proposition 3.2, and taking into account the weak-regularity of $\sigma(u)$ and the admissibility condition of the pair $(x, x-\alpha-1)$, the regularity of $\sigma(u)$, i.e., $\alpha \neq -n$, $n \geq 1$ follows. Therefore, $\mu \neq -n-(1/2)$, $n \geq 0$. Thus, $u(\mu)$ will be a semiclassical linear functional of class one. More precisely, it is the generalized Hermite functional denoted $\mathcal{H}(\mu)$ and $\sigma(u)$ is the Laguerre linear functional [1, 2, 4]. □

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