NONLINEARLY DEGENERATE WAVE EQUATION $v_{tt} = c(|v|^{s-1}v)_{xx}$

by

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Abstract

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It is well known that the generalized solutions for the Cauchy problem (1.4)-(1.5) are also the solutions of the nonlinearly degenerate wave equation $v_{tt} = c(|v|^{s-1}v)_{xx}$ with the initial data $v_0(x)$. In this paper, we first study the strong and weak entropies of system (1.4), then the H^{-1} compactness of $\eta(v^\varepsilon, u^\varepsilon)_t + q(v^\varepsilon, u^\varepsilon)_x$ of these entropy-entropy flux pairs with respect to the viscosity solutions of the Cauchy problem (1.7)-(1.5). Finally, suppose that for fixed point (x, t), the support set of the Young measure $\nu_{x,t}$ is concentrated on either the region $v \geq 0$ or the region $v \leq 0$, then $\nu_{x,t}$ must be a Dirac measure by using the theory of compensated compactness coupled with the kinetic formulation by **Lions**, **Perthame**, **Souganidis** and **Tadmor** [LPS, LPT].

Key words: Wave Equation, Young Measure, Compensated Compactness, Kinetic Formulation.

Resumen

Es bien conocido que las soluciones generalizadas para el problema de Cauchy (1.4)-(1.5) son también soluciones de la ecuación de ondas nolinealmente degenerada $v_{tt} = c(|v|^{s-1}v)_{xx}$ con valor inicial $v_0(x)$. En éste artículo, se empieza estudiando las entropías fuertes y débiles del sistema (1.4), luego la H^{-1} compacidad de $\eta(v^\varepsilon, u^\varepsilon)_t + q(v^\varepsilon, u^\varepsilon)_x$ de los pares de flujo entropía-entropía con respecto a las soluciones viscosas del problema de Cauchy (1.7)-(1.5). Finalmente, suponemos que para el punto (x, t) fijo, el soporte del conjunto de la medida de Young $\nu_{x,t}$ se concentra sobre la región $v \geq 0$ o la región $v \leq 0$, entonces $\nu_{x,t}$ debe ser una

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una medida de Dirac con el uso de la teoría de compacidad compensada y la formulación cinética de **Lions**, **Perthame**, **Souganidis** y **Tadmor** [LPS, LPT].

Palabras clave: Ecuación de Ondas, Medida de Young,Compacidad Compensada,Formulación Cinética.

1. Introduction

A function $v(x,t) \in L^\infty$ is called a generalized solution of the nonlinearly degenerate wave equation

$$v_{tt} = c(|v|^{s-1}v)_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

with initial data

$$(v, v_t)|_{t=0} = (v_0(x), v_1(x)), \quad -\infty < x < \infty \tag{1.2}$$

if for any test function $\phi \in C_0^\infty(R \times R^+)$,

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty v\phi(x,t)_{tt} - c|v|^{s-1}v\phi(x,t)_{xx}dxdt \\ + \int_{-\infty}^\infty v_0(x)\phi(x,0)_t - v_1(x)\phi(x,0)dx = 0,\end{aligned} \tag{1.3}$$

where $s > 1, c = \frac{\theta^2}{s} > 0$ and $\theta = -\frac{s+1}{2} < 0$ are constants.

A pair of functions $(v(x,t), u(x,t))$ is called a generalized solution of the following system

$$v_t - u_x = 0, \quad u_t - f(v)_x = 0 \tag{1.4}$$

with bounded initial date

$$(v, u)|_{t=0} = (v_0(x), u_0(x)) \tag{1.5}$$

if for any test function $\phi_i(x,t) \in C_0^\infty(R \times R^+)$, i=1,2,

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty v\phi(x,t)_{1t} - u\phi(x,t)_{1x}dxdt \\ \qquad + \int_{-\infty}^\infty v_0(x)\phi_1(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty u\phi(x,t)_{2t} - f(v)\phi(x,t)_{2x}dxdt \\ \qquad + \int_{-\infty}^\infty u_0(x)\phi_2(x,0)dx = 0. \end{cases} \tag{1.6}$$

It is obvious that the generalized solutions of the Cauchy problem (1.4)-(1.5) are also the solutions of the Cauchy problem (1.1)-(1.2) if we specially choose that $\phi_1 = \phi_t, \phi_2 = \phi_x$ and $v_1(x) = u_0'(x)$ in (1.6).

In this paper, we first study the strong and weak entropies of system (1.4), then the H^{-1} compactness of $\eta(v^\varepsilon, u^\varepsilon)_t + q(v^\varepsilon, u^\varepsilon)_x$ of these entropy-entropy flux pairs with respect to the viscosity solutions of the following parabolic system

$$v_t - u_x = \varepsilon v_{xx}, \quad u_t - f(v)_x = \varepsilon u_{xx} \tag{1.7}$$

with the initial date (1.5), where $f(v) = c|v|^{s-1}v$. Finally, suppose that for fixed point (x,t), the support set of the Young measure $\nu_{x,t}$ is concentrated on either the region $v \geq 0$ or the region $v \leq 0$, then we prove that $\nu_{x,t}$ must be a Dirac measure by using the theory of compensated compactness coupled with the kinetic formulation by **Lions**, **Perthame**, **Souganidis** and **Tadmor** [LPS, LPT]. Thus the limit $v(x,t)$ of $v^\varepsilon(x,t)$ is a solution of the wave equation (1.1).

2. Strong and weak entropies

A pair of smooth functions $(\eta(v,u), q(v,u))$ is called a pair of entropy-entropy flux of system (1.4) if $(\eta(v,u), q(v,u))$ satisfies

$$q_u = -\eta_v, \quad q_v = -\theta^2|v|^{s-1}\eta_u. \tag{2.1}$$

Eliminating the q from (2.1), we have the following entropy equation of system (1.4).

$$\eta_{vv} = \theta^2|v|^{s-1}\eta_{uu}. \tag{2.2}$$

An entropy $\eta(v,u)$ of system (1.4) is called a strong entropy if $\eta(0,u) \neq 0$; otherwise, it called a weak entropy.

Lemma 1. *Let $\bar\eta(v,u)$ satisfy entropy equation (2.2) in the region $v > 0$ and $\bar q(v,u)$ be the entropy flux. Then*

I. $\bar\eta(v,u)$ has an even extension and $\bar q(v,u)$ a odd extension if $\bar\eta(0,u) \neq 0$ and $\bar q(0,u) = 0$;

II. $\bar\eta(v,u)$ has a odd extension and $\bar q(v,u)$ an even extension if $\bar\eta(0,u) = 0$; i.e.,

$$\begin{cases} \eta(v,u) = \bar\eta(|v|,u), & \text{if } v \neq 0, \\ \eta(0,u) = \bar\eta(0,u), & \text{if } v = 0 \\ q(v,u) = (\text{ sgn } v)\bar q(|v|,u), & \text{if } v \neq 0, \\ q(0,u) = 0, & \text{if } v = 0 \end{cases} \tag{2.3}$$

or

$$\begin{cases} \eta(v,u) = (\text{ sgn } v)\bar\eta(|v|,u), & \text{if } v \neq 0, \\ \eta(0,u) = 0, & \text{if } v = 0, \\ q(v,u) = \bar q(|v|,u), & \text{if } v \neq 0, \\ q(0,u) = \bar q(0,u), & \text{if } v = 0 \end{cases} \tag{2.4}$$

satisfies system (2.1) on $v \neq 0$;

III. if $\eta_v(0,u) = 0$, then there exists an entropy flux associated to η such that $q(0,u) = 0$.

Proof of Lemma 1. Part I and Par II are easy to be proved from the equations in system (2.1). About the proof of Part III, from the first equation in (2.1), we have that

$$q(v,u) = -\int_0^u \eta_v(v,\tau)d\tau + h(v) \qquad (2.5)$$

and so

$$\begin{aligned} q_v(v,u) &= -\textstyle\int_0^u \eta_{vv}(v,\tau)d\tau + h'(v) \\ &= -\textstyle\int_0^u \theta^2|v|^{s-1}\eta_{uu}(v,\tau)d\tau + h'(v) \qquad (2.6) \\ &= -\theta^2|v|^{s-1}\eta_u(v,u) + \theta^2|v|^{s-1}\eta_u(v,0) + h'(v). \end{aligned}$$

Let $h(v) = -\int_0^v \theta^2|\tau|^{s-1}\eta_u(\tau,0)d\tau$ in (2.5). Then clearly q given by (2.5) is an entropy flux associated to η and satisfies $q(0,u) = 0$.

Consider (2.2) in the region $v > 0$ with the following initial conditions

$$\bar\eta(0,u) = d_1^0 f_1(u), \quad \bar\eta_v(0,u) = 0, \qquad (2.7)$$

where $d_1^0 = \int_{-1}^1 (1-\tau^2)^\lambda d\tau$. Then an entropy of (2.2) with (2.7) in the region $v > 0$ is

$$\bar\eta_1^0(v,u) = \int_{-\infty}^{\infty} f_1(\xi)G_1^0(v,\xi-u)d\xi, \qquad (2.8)$$

where the fundamental solution

$$G_1^0(v,u-\xi) = v(v^{s+1} - (\xi-u)^2)_+^\lambda \qquad (2.9)$$

the notation $x_+ = \max(0,x)$ and $\lambda = -\frac{s+3}{2(s+1)} \in (-1,0)$. The entropy flux $\bar q_1^0(v,u)$ associated with $\bar\eta_1^0(v,u)$ in the region $v > 0$ is

$$\bar q_1^0(v,u) = \int_{-\infty}^{\infty} f_1(\xi)\theta\frac{\xi-u}{v}G(v,\xi-u)d\xi. \qquad (2.10)$$

Similarly, an entropy of (2.2) with initial conditions

$$\bar\eta(0,u) = 0, \quad \bar\eta_v(0,u) = d_2^0 f_2(u), \qquad (2.11)$$

in the region $v > 0$ is

$$\bar\eta_2^0(v,u) = \int_{-\infty}^{\infty} f_2(\xi)G_2^0(v,\xi-u)d\xi, \qquad (2.12)$$

where the fundamental solution

$$G_2^0(v,u-\xi) = (v^{s+1} - (\xi-u)^2)_+^\mu \qquad (2.13)$$

and $d_2^0 = \int_{-1}^1 (1-\tau^2)^\mu d\tau, \mu = -\lambda - 1 \in (-1,0)$. The entropy flux $\bar q_2^0(v,u)$ associated with $\bar\eta_2^0(v,u)$ in the region $v > 0$ is

$$\bar q_2^0(v,u) = \int_{-\infty}^{\infty} f_2(\xi)[\theta\frac{\xi-u}{v}G_2^0(v,\xi-u) + \int_0^v G_2^0(y,\xi-u)dy]d\xi. \qquad (2.14)$$

We can also use the "exterior" of G_1 as suggested in [JPP, LPS, LPT] to get following functions satisfying system (2.1) in the region $v > 0$:

$$\bar\eta_1^\pm(v,u) = \int_{-\infty}^{\infty} f_1(\xi)G_1^\pm(v,\xi-u)d\xi \qquad (2.15)$$

with initial conditions

$$\begin{aligned} &\bar\eta_1^\pm(0,u) = d_1^\pm f_1(u), \\ &\bar\eta_{1v}^\pm(0,u) = \mp\theta\int_0^\infty f_1'(\pm y+u)y^{2\lambda+1}dy \end{aligned} \qquad (2.16)$$

where $f_1(u)$ has a compact support set in $(-\infty,\infty)$, $d_1^\pm = \int_0^\infty \tau^\lambda(\tau+2)^\lambda d\tau$ and the fundamental solutions

$$G_1^+(v,u-\xi) = v(\xi-(u+v^{\frac{s+1}{2}})_+^\lambda(\xi-(u-v^{\frac{s+1}{2}})^\lambda, \qquad (2.17)$$

$$G_1^-(v,u-\xi) = v(u-v^{\frac{s+1}{2}}-\xi)_+^\lambda(u+v^{\frac{s+1}{2}}-\xi)^\lambda. \qquad (2.18)$$

Similarly

$$\bar\eta_2^\pm(v,u) = \int_{-\infty}^{\infty} f_2(\xi)G_2^\pm(v,\xi-u)d\xi \qquad (2.19)$$

with initial conditions

$$\bar\eta_2^\pm(0,u) = \int_0^\infty f_2(\pm y+u)y^{2\mu}dy, \ \bar\eta_{2v}^\pm(0,u) = d_2^\pm f_2(u) \qquad (2.20)$$

where $f_2(u)$ has a compact support set in $(-\infty,\infty)$, $d_2^\pm = \frac{1-s}{2}\int_0^\infty \tau^\mu(\tau+2)^{\mu-1}d\tau$ and the fundamental solutions

$$G_2^+(v,u-\xi) = (\xi-(u+v^{\frac{s+1}{2}})_+^\mu(\xi-(u-v^{\frac{s+1}{2}})^\mu, \qquad (2.21)$$

$$G_2^-(v,u-\xi) = (u-v^{\frac{s+1}{2}}-\xi)_+^\mu(u+v^{\frac{s+1}{2}}-\xi)^\mu. \qquad (2.22)$$

Theorem 2.

A. *The pair of functions*

$$\begin{cases} \eta_1^0(v,u) = \bar\eta_1^0(|v|,u), & \text{if } v \neq 0, \\ \eta_1^0(0,u) = d_1^0 f_1(u), & \text{if } v = 0 \\ q_1^0(v,u) = (\operatorname{sgn} v)\bar q_1^0(|v|,u), & \text{if } v \neq 0, \\ q_1^0(0,u) = 0, & \text{if } v = 0 \end{cases} \qquad (2.23)$$

is an entropy-entropy flux pair of system (1.4).

B. *System (1.4) has the following entropies:*

$$\begin{cases} \eta_1(v,u) = (\operatorname{sgn} v)(\bar\eta_1^+(|v|,u) - \bar\eta_1^-(|v|,u)), \text{ if } v \neq 0 \\ \eta_1(0,u) = 0; \end{cases} \qquad (2.24)$$

$$\begin{cases} \eta_2^0(v,u) = \bar\eta_2^+(|v|,u) - \bar\eta_2^-(|v|,u), & \text{if } v \neq 0 \\ \eta_2^0(0,u) = \int_0^\infty (f_2(y+u) - f_2(-y+u))y^{2\mu}dy; \end{cases} \qquad (2.25)$$

$$\begin{cases} \eta_2^+(v,u) = C\bar\eta_2^0(|v|,u) + \bar\eta_2^+(|v|,u), & \text{if } v \neq 0 \\ \eta_2^+(0,u) = \int_0^\infty f_2(y+u)y^{2\mu}dy; \end{cases} \qquad (2.26)$$

$$\begin{cases}\eta_2^-(v,u) = C\bar{\eta}_2^0(|v|,u) + \bar{\eta}_2^-(|v|,u), & \text{if } v \neq 0 \\ \eta_2^+(0,u) = \int_0^\infty f_2(-y+u)y^{2\mu}dy, \end{cases} \tag{2.27}$$

where

$$C = \frac{(s-1)\int_0^\infty (\tau+2)^{\mu-1}\tau^\mu d\tau}{2\int_{-1}^1 (1-\tau^2)^\mu d\tau} > 0. \tag{2.28}$$

Proof of Theorem 2. Since the results in Lemma 1, to prove Part (A), it is sufficient to prove that $(\bar{\eta}_1^0(v,u), \bar{q}_1^0(v,u))$ satisfies the entropy-entropy flux system (2.1) on $v \geq 0$ and $\bar{q}_1^0(0,u) = 0$. In fact, letting $\xi = u + v^{\frac{s+1}{2}}\tau, w = u + v^{\frac{s+1}{2}}$ and $z = u - v^{\frac{s+1}{2}}$, we have on $v > 0$

$$\begin{aligned}\bar{\eta}_1^0(v,u) &= \int_{-\infty}^\infty f_1(\xi)G_1^0(v,\xi-u)d\xi \\ &= \int_z^w f_1(\xi)v(w-\xi)^\lambda(\xi-z)^\lambda d\xi \\ &= \int_{-1}^1 f_1(u+v^{\frac{s+1}{2}}\tau)v(v^{\frac{s+1}{2}})^{(2\lambda+1)}(1-\tau^2)^\lambda d\tau \\ &= \int_{-1}^1 f_1(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^\lambda d\tau,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\bar{q}_1^0(v,u) &= \int_{-\infty}^\infty f_1(\xi)\theta\frac{\xi-u}{v}G_1^0(v,\xi-u)d\xi \\ &= \int_z^w f_1(\xi)\theta(\xi-u)(w-\xi)^\lambda(\xi-z)^\lambda d\xi \\ &= \int_{-1}^1 f_1(u+v^{\frac{s+1}{2}}\tau)\theta v^{\frac{s+1}{2}}\tau(v^{\frac{s+1}{2}})^{(2\lambda+1)}(1-\tau^2)^\lambda d\tau \\ &= \int_{-1}^1 f_1(u+v^{\frac{s+1}{2}}\tau)\theta v^{\frac{s-1}{2}}\tau(1-\tau^2)^\lambda d\tau.\end{aligned} \tag{2.30}$$

So, $\bar{q}_1^0(0,u) = 0$ and $\bar{q}_{1u}^0(v,u) = -\bar{\eta}_{1v}^0(v,u)$ on $v \geq$. To prove the second equation in (2.1), letting

$$h(\tau) = \int_{-1}^\tau \xi(1-\xi^2)^\lambda d\xi = -\frac{s+1}{s-1}(1-\tau^2)^{\lambda+1}, \tag{2.31}$$

we have

$$\begin{aligned}\bar{q}_1^0(v,u) &= \int_{-1}^1 f_1(u+v^{\frac{s+1}{2}}\tau)\theta v^{\frac{s-1}{2}}dh(\tau) \\ &= \int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)\theta v^s h(\tau)d\tau\end{aligned} \tag{2.32}$$

and thus

$$\begin{aligned}\bar{q}_{1v}^0 &= -\int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)\theta s v^{s-1}h(\tau)d\tau \\ &\quad + \int_{-1}^1 f_1''(u+v^{\frac{s+1}{2}}\tau)\theta^2 v^{\frac{3s-1}{2}}\tau h(\tau)d\tau \\ &= I + II,\end{aligned} \tag{2.33}$$

where

$$I = -\theta^2\frac{2s}{s-1}v^{s-1}\int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^{\lambda+1}d\tau, \tag{2.34}$$

and

$$\begin{aligned}II &= \theta^2 v^{s-1}\int_{-1}^1 \tau h(\tau)d(f_1'(u+v^{\frac{s+1}{2}}\tau)) \\ &= -\theta^2 v^{s-1}\int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)(h(\tau)+\tau^2(1-\tau^2)^\lambda)d\tau \\ &= \theta^2\frac{2s}{s-1}v^{s-1}\int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^{\lambda+1}d\tau \\ &\quad - \theta^2 v^{s-1}\int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^\lambda d\tau.\end{aligned} \tag{2.35}$$

Therefore

$$\begin{aligned}\bar{q}_{1v}^0 &= -\theta^2 v^{s-1}\int_{-1}^1 f_1'(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^\lambda d\tau \\ &= -\theta^2 v^{s-1}\bar{\eta}_{1u}^0\end{aligned} \tag{2.36}$$

on $v \geq 0$. Part (A) is proved.

To prove that $\eta_1(v,u)$ given by (2.24) is an entropy, we rewrite $\bar{\eta}_1^+$ in the region $v > 0$ by

$$\begin{aligned}\bar{\eta}_1^+ &= \int_w^\infty f_1(\xi)v(\xi-w)^\lambda(\xi-z)^\lambda d\xi \ (\xi - w = y) \\ &= \int_0^\infty f_1(y+u+v^{\frac{s+1}{2}})vy^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy \ (y = v^{\frac{s+1}{2}}\tau) \\ &= \int_0^\infty f_1(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\lambda(\tau+2)^\lambda d\tau.\end{aligned} \tag{2.37}$$

Similarly

$$\bar{\eta}_1^- = \int_0^\infty f_1(-v^{\frac{s+1}{2}}\tau+u-v^{\frac{s+1}{2}})\tau^\lambda(\tau+2)^\lambda d\tau. \tag{2.38}$$

Thus

$$\eta_1(0,u) = \bar{\eta}_1^+(0,u) - \bar{\eta}_1^-(0,u) = 0. \tag{2.39}$$

To prove that $\bar{\eta}_1^+$ satisfies equation (2.2) in the region $v>0$, we use the second equation in (2.37) to get

$$\begin{aligned}\bar{\eta}_{1v}^+ =&\int_0^\infty f_1(y+u+v^{\frac{s+1}{2}})y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\\ &-\theta\int_0^\infty f_1'(y+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}}y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\\ &-2\lambda\theta\int_0^\infty f_1(y+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}}y^\lambda(y+2v^{\frac{s+1}{2}})^{\lambda-1}dy\\ =&\, I_1+I_2,\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}I_1=&\int_0^\infty f_1(y+u+v^{\frac{s+1}{2}})y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\\ &-\theta\int_0^\infty f_1'(y+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}}y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy,\end{aligned} \tag{2.41}$$

$$\begin{aligned}I_{1v}=&(\theta^2-\theta)\int_0^\infty f_1'(y+u+v^{\frac{s+1}{2}})v^{\frac{s-1}{2}}y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\\ &-2\lambda\theta\int_0^\infty f_1(y+u+v^{\frac{s+1}{2}})v^{\frac{s-1}{2}}y^\lambda(y+2v^{\frac{s+1}{2}})^{\lambda-1}dy\\ &+2\lambda\theta^2\int_0^\infty f_1'(y+u+v^{\frac{s+1}{2}})v^s y^\lambda(y+2v^{\frac{s+1}{2}})^{\lambda-1}dy\\ &+\theta^2\int_0^\infty f_1''(y+u+v^{\frac{s+1}{2}})v^s y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\\ =&(\theta^2-\theta)\int_0^\infty f_1'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{\frac{s-1}{2}-1}\tau^\lambda(\tau+2)^\lambda d\tau\\ &-2\lambda\theta\int_0^\infty f_1(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{-2}\tau^\lambda(\tau+2)^{\lambda-1}d\tau\\ &+2\lambda\theta^2\int_0^\infty f_1'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{\frac{s-1}{2}-1}\tau^\lambda(\tau+2)^{\lambda-1}d\tau\\ &+\theta^2\int_0^\infty f_1''(y+u+v^{\frac{s+1}{2}})v^s y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}I_2=&-2\lambda\theta\int_0^\infty f_1(y+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}}y^\lambda(y+2v^{\frac{s+1}{2}})^{\lambda-1}dy\\ =&-2\lambda\theta\int_0^\infty f_1(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{-1}\tau^\lambda(\tau+2)^{\lambda-1}d\tau,\end{aligned} \tag{2.43}$$

$$\begin{aligned}I_{2v}=&2\lambda\theta\int_0^\infty f_1(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{-2}\tau^\lambda(\tau+2)^{\lambda-1}d\tau\\ &+2\lambda\theta^2\int_0^\infty f_1'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}-2}(\tau+1)\tau^\lambda(\tau+2)^{\lambda-1}d\tau\\ =&2\lambda\theta\int_0^\infty f_1(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{-2}\tau^\lambda(\tau+2)^{\lambda-1}d\tau\\ &+2\lambda\theta^2\int_0^\infty f_1'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}-2}(\tau+1)\tau^\lambda(\tau+2)^{\lambda}d\tau\\ &-2\lambda\theta^2\int_0^\infty f_1'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v^{\frac{s+1}{2}-2}(\tau+1)\tau^\lambda(\tau+2)^{\lambda-1}d\tau.\end{aligned} \tag{2.44}$$

Since $\theta^2-\theta=-2\lambda\theta^2$, we have by combining (2.40), (2.42) and (2.40) that

$$\begin{aligned}\bar{\eta}_{1vv}^+=&\theta^2\int_0^\infty f_1''(y+u+v^{\frac{s+1}{2}})v^s y^\lambda(y+2v^{\frac{s+1}{2}})^\lambda dy\\ =&\theta^2 v^{s-1}\bar{\eta}_{1uu}^+.\end{aligned} \tag{2.45}$$

Similarly, we can prove that

$$\bar{\eta}_{1vv}^- = \theta^2 v^{s-1}\bar{\eta}_{1uu}^- \tag{2.46}$$

in the region $v>0$. Thus $\eta_1(v,u)$ given by (2.24) is an entropy of system (1.4).

About the functions given by (2.25)-(2.27), we only provide the proof for $\eta_2^+(v,u)$. A similar treatment gives the proof for $\eta_2^0(v,u),\eta_2^-(v,u)$.

Using Lemma 1, to prove that $\eta_2^+(v,u)$ is an entropy of system (1.4), it is sufficient to prove that $\bar{\eta}_2^0(v,u),\bar{\eta}_2^+(v,u)$ satisfy entropy equation (2.2) in the region of $v>0$ and $C\bar{\eta}_{2v}^0(0,u)+\bar{\eta}_{2v}^+(0,u)=0$.

We may rewrite $\bar{\eta}_2^0(v,u)$ by

$$\bar{\eta}_2^0(v,u)=\int_{-1}^1 f_2(u+v^{\frac{s+1}{2}}\tau)v(1-\tau^2)^\mu d\tau. \tag{2.47}$$

Then

$$\begin{aligned}\bar{\eta}_{2v}^0(v,u)=&\int_{-1}^1 f_2(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^\mu d\tau\\ &-\theta\int_{-1}^1 f_2'(u+v^{\frac{s+1}{2}}\tau)v^{\frac{s+1}{2}}\tau(1-\tau^2)^\mu d\tau\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}\bar{\eta}_{2vv}^0(v,u)=&(\theta^2-\theta)\int_{-1}^1 f_2'(u+v^{\frac{s+1}{2}}\tau)v^{\frac{s-1}{2}}\tau(1-\tau^2)^\mu d\tau\\ &+\theta^2\int_{-1}^1 f_2''(u+v^{\frac{s+1}{2}}\tau)v^s\tau^2(1-\tau^2)^\mu d\tau.\end{aligned} \tag{2.49}$$

Letting

$$h_1(\tau) = \int_{-1}^{\tau} \xi(1-\xi^2)^\mu d\xi = -\frac{s+1}{s+3}(1-\tau^2)^{\mu+1}, \quad (2.50)$$

we have that

$$\begin{aligned}(\theta^2-\theta)&\int_{-1}^{1} f_2'(u+v^{\frac{s+1}{2}}\tau)v^{\frac{s-1}{2}}\tau(1-\tau^2)^\mu d\tau\\ &= -(\theta^2-\theta)v^{\frac{s-1}{2}}\int_{-1}^{1} f_2''(u+v^{\frac{s+1}{2}}\tau)h_1(\tau)v^{\frac{s+1}{2}}d\tau\\ &= \theta^2 v^s\int_{-1}^{1} f_2''(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^{\mu+1}d\tau.\end{aligned} \quad (2.51)$$

Thus we get by (2.49) and (2.51) that

$$\begin{aligned}\bar\eta^0_{2vv}(v,u) &= \theta^2 v^s\int_{-1}^{1} f_2''(u+v^{\frac{s+1}{2}}\tau)(1-\tau^2)^\mu d\tau\\ &= \theta^2 v^{s-1}\bar\eta^0_{2uu}(v,u).\end{aligned} \quad (2.52)$$

Similarly we may rewrite $\bar\eta_2^+(v,u)$ by

$$\begin{aligned}\bar\eta_2^+ &= \int_w^\infty f_2(\xi)v(\xi-w)^\mu(\xi-z)^\mu d\xi \quad (\xi-w=y)\\ &= \int_0^\infty f_2(y+u+v^{\frac{s+1}{2}})y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy\\ &\qquad\qquad (y=v^{\frac{s+1}{2}}\tau)\\ &= \int_0^\infty f_2(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})v\tau^\mu(\tau+2)^\mu d\tau.\end{aligned} \quad (2.53)$$

To prove that $\bar\eta_2^+$ satisfies equation (2.2) in the region $v>0$, we use the second equation in (2.53) to get

$$\begin{aligned}\bar\eta^+_{2v} &= -\theta\int_0^\infty f_2'(y+u+v^{\frac{s+1}{2}})v^{\frac{s-1}{2}}y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy\\ &- 2\theta\mu v^{\frac{s-1}{2}}\int_0^\infty f_2(y+u+v^{\frac{s+1}{2}})y^\mu(y+2v^{\frac{s+1}{2}})^{\mu-1}dy\\ &= I_3+I_4,\end{aligned} \quad (2.54)$$

where

$$\begin{aligned}I_{3v} &= \theta^2\int_0^\infty f_2''(y+u+v^{\frac{s+1}{2}})v^{s-1}y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy + \frac{s+1}{2}\frac{s-1}{2}v^{\frac{s-1}{2}-1}\int_0^\infty f_2'(y+u+v^{\frac{s+1}{2}})y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy\\ &\quad + 2\mu\theta^2 v^{s-1}\int_0^\infty f_2'(y+u+v^{\frac{s+1}{2}})y^\mu(y+2v^{\frac{s+1}{2}})^{\mu-1}dy\\ &= \theta^2\int_0^\infty f_2''(y+u+v^{\frac{s+1}{2}})v^{s-1}y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy + \frac{s+1}{2}\frac{s-1}{2}v^{\frac{s-1}{2}-1}\int_0^\infty f_2'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\mu(\tau+2)^\mu(v^{\frac{s+1}{2}})^{2\mu+1}d\tau\\ &\quad + 2\mu\theta^2 v^{s-1}\int_0^\infty f_2'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\mu(\tau+2)^{\mu-1}(v^{\frac{s+1}{2}})^{2\mu}d\tau\\ &= \theta^2\int_0^\infty f_2''(y+u+v^{\frac{s+1}{2}})v^{s-1}y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy\\ &\quad + \frac{s+1}{2}\frac{s-1}{2}v^{\frac{s-1}{2}}\int_0^\infty f_2'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\mu(\tau+2)^{\mu-1}(\tau+1)d\tau\end{aligned} \quad (2.55)$$

since $2\mu\theta^2 = -\frac{s+1}{2}\frac{s-1}{2}$ and

$$\begin{aligned}I_4 &= -2\theta\mu v^{\frac{s-1}{2}}\int_0^\infty f_2(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\mu(\tau+2)^{\mu-1}(v^{\frac{s+1}{2}})^{2\mu}d\tau\\ &= \frac{1-s}{2}\int_0^\infty f_2(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\mu(\tau+2)^{\mu-1}d\tau,\end{aligned} \quad (2.56)$$

$$I_{4v} = \frac{s+1}{2}\frac{s-1}{2}v^{\frac{s-1}{2}}\int_0^\infty f_2'(v^{\frac{s+1}{2}}\tau+u+v^{\frac{s+1}{2}})\tau^\mu(\tau+2)^{\mu-1}(\tau+1)d\tau, \quad (2.57)$$

thus

$$\bar\eta^+_{2vv} = \theta^2\int_0^\infty f_2''(y+u+v^{\frac{s+1}{2}})v^{s-1}y^\mu(y+2v^{\frac{s+1}{2}})^\mu dy = \theta^2 v^{s-1}\bar\eta^+_{2uu}, \quad (2.58)$$

which implies that $\bar\eta_2^+$ satisfies the entropy equation (2.2) in the region of $v>0$.

It is obvious that

$$\bar{\eta}^0_{2v}(0,u) = f_2(u)\int_{-1}^{1}(1-\tau^2)^{\mu}d\tau,$$
$$\bar{\eta}^+_{2v}(0,u) = \frac{1-s}{2}f_2(u)\int_0^{\infty}\tau^{\mu}(\tau+2)^{\mu-1}d\tau \tag{2.59}$$

from (2.48),(2.54) and (2.56). Thus $\eta^0_{2v}(0,u) = 0$ and the proof of Theorem 2 is ended.

3. Compactness

In this section, we study the H^{-1} compactness of $\eta(v^\varepsilon,u^\varepsilon)_t+q(v^\varepsilon,u^\varepsilon)_x$, of these entropy-entropy flux pairs given in Section 2, with respect to the viscosity solutions of the Cauchy problem (1.7)-(1.5).

By using the invariant region theory by **Chueh, Conley** and **Smoller** [CCS], we can obtain the following L^∞ estimate on the solutions $(v^\varepsilon,u^\varepsilon)$ of the Cauchy problem (1.7)-(1.5):

$$|v^\varepsilon|\le M,\quad |u^\varepsilon|\le M, \tag{3.1}$$

where M is a positive constant, which is independent of ε, but depends on the L^∞ bound of the initial data (1.5).

In this section we obtain the main results as follows:

Theorem 3.

$$v^\varepsilon(x,t)_t - u^\varepsilon(x,t)_x \tag{3.2}$$

is compact in $H^{-1}_{loc}(R\times R^+)$ *and*

$$\eta^s(v^\varepsilon(x,t),u^\varepsilon(x,t))_t + q^s(v^\varepsilon(x,t),u^\varepsilon(x,t))_x \tag{3.3}$$

is compact in $H^{-1}_{loc}(R\times R^+)$, *where* $\eta^s(v,u)=\eta^0_1(v,u)$ *or* $\eta^0_2(v,u),\eta^+_2(v,u)$ *and* $\eta^-_2(v,u)$.

Theorem 4. *Let $\nu_{x,t}$ be the family of positive probability measures with respect to the viscosity solutions $(v^\varepsilon,u^\varepsilon)$ of the Cauchy problem (1.7) and (1.5). Suppose, for fixed (x,t), the support set of the Young measure $\nu_{x,t}$ is concentrated on either $v\ge 0$ or $v\le 0$. Then for this point (x,t), the Young measure $\nu_{x,t}$ must be a Dirac measure.*

Proof of Theorem 3. For simplicity, we omit the superscript ε. We multiply (1.7) by $(\eta^\star_v,\eta^\star_u)$ to obtain the boundedness of

$$\varepsilon(v_x,u_x)\cdot\nabla^2\eta^\star(v,u)\cdot(v_x,u_x)^T \tag{3.4}$$

in $L^1_{loc}(R\times R^+)$, where $\eta^* = \frac{u^2}{2}+\frac{s+1}{4s}|v|^{s+1}$ is a convex entropy of system (1.4). Then it follows that

$$\varepsilon u_x^2 + \varepsilon\theta^2|v|^{s-1}v_x^2 \tag{3.5}$$

are bounded in $L^1_{loc}(R\times R^+)$.

We multiply (1.7) by v and then by a test function ϕ, where $\phi\in C_0^\infty(R\times R^+)$ satisfies $\phi_K=1, 0\le\phi\le 1$ and $S=\operatorname{supp}\phi$ for an arbitrary compact set $K\subset S\subset R\times R^+$. Then, we have that

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty \varepsilon v_x^2\phi\,dxdt &= \int_0^\infty\int_{-\infty}^\infty(\frac{v^2}{2}\phi_t+\varepsilon\frac{v^2}{2}\phi_{xx})dxdt\\ &+\int_0^\infty\int_{-\infty}^\infty vu_x\phi\,dxdt\le M(\phi)\\ &+\left(\int_0^\infty\int_{-\infty}^\infty v^2\phi\,dxdt\right)^{\frac12}\left(\int_0^\infty\int_{-\infty}^\infty u_x^2\phi\,dxdt\right)^{\frac12}\end{aligned} \tag{3.6}$$

and hence by (3.5),

$$\varepsilon^{\frac32}v_x^2\quad\text{are bounded in } L^1_{loc}(R\times R^+), \tag{3.7}$$

which combining with the first equation in (1.7) imply the proof of (3.2).

To prove (3.3), it is sufficient to prove the following

Lemma 5. *If*

$$\eta_v(0,u)=0,\quad \frac{\partial^i\eta(v,u)}{\partial u^i}\quad i=0,1,2,3,$$

are bounded in $0\le|v|\le M, |u|\le M$, *then*

$$\eta(v^\varepsilon(x,t),u^\varepsilon(x,t))_t+q(v^\varepsilon(x,t),u^\varepsilon(x,t))_x$$

is compact in $H^{-1}_{loc}(R\times R^+)$.

Proof of Lemma 5. Using entropy equation (2.2) and the condition $\eta_v(0,u)=0$, we have

$$\eta_v(v,u)=\int_0^v\theta^2|\xi|^{s-1}\eta_{uu}(\xi,u)d\xi \tag{3.8}$$

and

$$\eta_{vu}(v,u)=\int_0^v\theta^2|\xi|^{s-1}\eta_{uuu}(\xi,u)d\xi. \tag{3.9}$$

Thus

$$\begin{aligned}|\eta_v(v,u)|&\le M\int_0^{|v|}\theta^2\xi^{s-1}d\xi\le M|v|^s\\ &\le M_1|v|^{\frac{s-1}{2}},\quad |\eta_{vu}(v,u)|\le M_1|v|^{\frac{s-1}{2}},\end{aligned} \tag{3.10}$$

where M, M_1 are positive constants.

Multiplying system (1.7) by $(\eta(v,u)_v,\eta(v,u)_u)$, we have

$$\begin{aligned}\eta(v\varepsilon,u^\varepsilon)_t+q(v^\varepsilon,u^\varepsilon)_x &= \varepsilon\eta(v^\varepsilon,u^\varepsilon)_{xx}-\varepsilon(\eta(v^\varepsilon,u^\varepsilon)_{vv}(v_x^\varepsilon)^2\\ &+2\eta(v^\varepsilon,u^\varepsilon)_{vu}v_x^\varepsilon u_x^\varepsilon+\eta(v^\varepsilon,u^\varepsilon)_{uu}(u_x^\varepsilon)^2).\end{aligned} \tag{3.11}$$

Using the first estimate in (3.10), (3.5) and the boundedness of η_u, we have the compactness of

$$\varepsilon\eta(v^\varepsilon, u^\varepsilon)_{xx} \quad \text{in} \quad H^{-1}_{loc}(R \times R^+), \tag{3.12}$$

as ε tends to zero. Using the second estimate in (3.10), (3.5), the boundedness of η_{uu} and the relation $\eta_{vv} = \theta^2|v|^{s-1}\eta_{uu}$, we have the boundedness of

$$\varepsilon(\eta(v^\varepsilon, u^\varepsilon)_{vv}(v^\varepsilon_x)^2 + 2\eta(v^\varepsilon, u^\varepsilon)_{vu}v^\varepsilon_x u^\varepsilon_x + \eta(v^\varepsilon, u^\varepsilon)_{uu}(u^\varepsilon_x)^2) \tag{3.13}$$

in $L^1_{loc}(R \times R^+)$, and hence the compactness in $W^{-1,\alpha}$ for some $\alpha \in (1, 2)$ by Sobolev's embedding Theorem.

Therefore the right-hand side of (3.11) is compact in $W^{-1,\alpha}_{loc}(R \times R^+)$ for some $\alpha \in (1, 2)$, but the left-hand side is bounded in $W^{-1,\infty}(R \times R^+)$. This implies the compactness of $\eta(v^\varepsilon, u^\varepsilon)_t + q(v^\varepsilon, u^\varepsilon)_x$ in $H^{-1}_{loc}(R\times R^+)$ by **Murat**'s Theorem [Mu], and hence the proof of Lemma 5. Theorem 3 is ended.

Proof of Theorem 4. I. Suppose $\operatorname{supp}\nu_{x,t} = 0$, then using the measure equation to the entropy-entropy flux pairs $(v, -u)$ and $(u, c|v|^{s-1}v)$, we get

$$< \nu_{x,t}, u >^2 = < \nu_{x,t}, u^2 >, \tag{3.14}$$

which inplies that $\nu_{x,t}$ is a Dirac measure and the support set is one point $(0, \bar{u})$.

II. Suppose, for fixed (x, t), the support set of the Young measure $\nu_{x,t}$ is concentrated on either $v \geq 0$ or $v \leq 0$, but not only on $v = 0$. Then clearly $< \nu_{x,t}, \eta^0_1 > \neq 0$. Using the measure equation in the theory of compensated compactness to the entropy-entropy flux pairs $(\eta^0_1(v, u), q^0_1(v, u))$, we get

$$\begin{aligned} &< |v|H(v, u, \xi) > < (\operatorname{sgn} v)(\xi' - u)H(v, u, \xi') > \\ &- < |v|H(v, u, \xi') >< (\operatorname{sgn} v)(\xi' - u)H(v, u, \xi) > \\ &= < (\xi' - \xi)vH(v, u, \xi)H(v, u, \xi') >, \end{aligned} \tag{3.15}$$

where we use the notation

$$< H(v, u, \xi) > = < \nu_{x,t}, H(v, u, \xi) >$$

and

$$H(v, u, \xi) = (|v|^{s+1} - (\xi - u)^2)^\lambda_+. \tag{3.16}$$

Let $w = u + |v|^{\frac{s+1}{2}}, z = u - |v|^{\frac{s+1}{2}}$ and $I = [z, w]$ for each $(v, u) \in \operatorname{supp}\nu_{x,t}$. Dividing (3.15) by

$$< |v|H(v, u, \xi) >< |v|H(v, u, \xi') >$$

and sending ξ' to ξ, we obtain

$$\frac{\partial}{\partial\xi}\left[\frac{<(\operatorname{sgn} v)(\xi - u)H(v, u, \xi)>}{<|v|H(v, u, \xi)>}\right] = \frac{< vH(v, u, \xi)^2 >}{<|v|H(v, u, \xi)>^2}. \tag{3.17}$$

Again using the measure equation between $(\eta^0_1(v, u), q^0_1(v, u))$ and $(v, -u)$, we get

$$\begin{aligned} &\theta<v><(\operatorname{sgn} v)(\xi - u)H(v, u, \xi)> \\ &\quad + <u><|v|H(v, u, \xi)> \\ &= \theta < (\xi - u)|v|H(v, u, \xi) > \\ &\quad + <u|v|H(v, u, \xi)>, \end{aligned} \tag{3.18}$$

which can be rewritten as

$$\begin{aligned} &\frac{\theta<v><(\operatorname{sgn} v)(\xi - u)H(v, u, \xi)> + <u><|v|H(v, u, \xi)>}{<|v|H(v, u, \xi)>} + <u \\ &= \theta\xi + (1 - \theta)\frac{< u|v|H(v, u, \xi) >}{< |v|H(v, u, \xi) >} \end{aligned} \tag{3.19}$$

in I. Differentiating (3.19) in ξ and combining the outcome with (3.17), we also get

$$\begin{aligned} &\frac{\partial}{\partial\xi}\left(\frac{<u|v|H(v, u, \xi)>}{<|v|H(v, u, \xi)>}\right) \\ &= \frac{\theta}{1 - \theta}\left(<v> \frac{<vH(v, u, \xi)^2>}{<|v|H(v, u, \xi)>^2} - 1\right) \leq 0 \end{aligned} \tag{3.20}$$

if the support set of the Young measure $\nu_{x,t}$ is concentrated on either $v \geq 0$ or $v \leq 0$. Following the steps given in Proposition II.1 in [LPS], we can end the proof of Theorem 4.

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