INVERSE FINITE-TYPE RELATIONS BETWEEN SEQUENCES OF POLYNOMIALS

By

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Abstract


Let \( \phi \) be a monic polynomial, with \( \deg \phi = t \geq 0 \). We say that there is a finite-type relation between two monic polynomial sequences \( \{B_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) with respect to \( \phi \), if there exists \( (s, r) \in \mathbb{N}^2, r \geq s \), such that

\[
\phi(x)Q_n(x) = \sum_{\nu = n-s}^{n+t} \lambda_{n,\nu} B_{\nu}(x), \quad n \geq s, \quad \text{with } \lambda_{r, r-s} \neq 0. \quad (*)
\]

The corresponding inverse finite-type relation of \((*)\) consists in a finite-type relation as follows:

\[
\Omega^*_s(x; n)B_n(x) = \sum_{\nu = n-t}^{n+s} \theta^*_n,\nu Q_{\nu}(x), \quad n \geq t, \quad \text{with } \theta^*_r, r \neq 0,
\]

where \( \deg \Omega^*_s(x; n) = s, n \geq t \). When the orthogonality of the two previous sequences is assumed, the inverse finite-type relation is always possible [11]. This work essentially studies the case when only the sequence \( \{B_n\}_{n \geq 0} \) is orthogonal. In fact, we find necessary and sufficient conditions leading to inverse finite-type relations. In particular, the structure relation characterizing semi-classical sequences is a special case of the general situation. Some examples will be analyzed.

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Resumen

Sea $\phi$ un polinomio mónico, con $\deg \phi = t \geq 0$. Decimos que hay relación de tipo finito entre dos sucesiones de polinomios mónicos $\{B_n\}_{n \geq 0}$ y $\{Q_n\}_{n \geq 0}$ con respecto a $\phi$, si existe $(s, r) \in \mathbb{N}^2$, $r \geq s$, tal que

$$\phi(x)Q_n(x) = \sum_{\nu = n-s}^{n+t} \lambda_{n,\nu} B_\nu(x), \quad n \geq s, \text{ con } \lambda_{r-r-s} \neq 0. \quad (*)$$

La correspondiente relación de tipo finito de $(*)$ consiste en una relación de tipo finito como sigue:

$$\Omega^*_n(x; n)B_n(x) = \sum_{\nu = n-t}^{n+s} \theta^*_n,\nu Q_\nu(x), \quad n \geq t, \text{ con } \theta^*_{r+t-r} \neq 0,$$

donde $\deg \Omega^*_n(x; n) = s, n \geq t$. Cuando se supone la ortogonalidad de las dos sucesiones previas, la relación de tipo finito inversa siempre es posible [11]. En este trabajo se estudia el caso en que solo la sucesión $\{B_n\}_{n \geq 0}$ es ortogonal. De hecho, encontramos condiciones necesarias y suficientes que conducen a relaciones de tipo finito inversas. En particular, la relación de estructura que caracteriza a las sucesiones semiclásicas es un caso especial de la situación general. Se estudian varios ejemplos.

Palabras clave: Relaciones de tipo finito, relaciones de recurrencia, polinomios ortogonales, polinomios semi-clásicos.

1. Introduction and background

Let $\mathbb{P}$ be the linear space of complex polynomials in one variable and $\mathbb{P}'$ its topological dual space. We denote by $\langle u, f \rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of $u$ with respect to the polynomial sequence $\{x^n\}_{n \geq 0}$.

We will introduce some useful operations in $\mathbb{P}'$. For any linear functional $u$ and any polynomial $h$, let $Du = u'$ and $hu$ be the linear functionals defined by duality

$$\langle u', f \rangle := -\langle u, f' \rangle, \quad f \in \mathbb{P},$$

$$\langle hu, f \rangle := \langle u, hf \rangle, \quad f, h \in \mathbb{P}.$$

Let $\{B_n\}_{n \geq 0}$ be a monic polynomial sequence (MPS), $\deg B_n = n$, $n \geq 0$, and $\{u_n\}_{n \geq 0}$ its dual sequence, $u_n \in \mathbb{P}'$, $n \geq 0$, defined by $\langle u_n, B_m \rangle := \delta_{n,m}$, $n, m \geq 0$, where $\delta_{n,m}$ is the Kronecker symbol.

Let recall the following results [11].

Lemma 1.1. For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent.

i) $\langle u, B_{m-1} \rangle \neq 0, \quad \langle u, B_m \rangle = 0, \quad n \geq m$.

ii) There exist $\lambda_{\nu} \in \mathbb{C}, 0 \leq \nu \leq m - 1, \lambda_{m-1} \neq 0$, such that $u = \sum_{\nu = 0}^{m-1} \lambda_{\nu} u_{\nu}$.

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n \geq 0}$ of the sequence $\{B_n^{[1]}\}_{n \geq 0}$, where $B_n^{[1]}(x) = (n + 1)^{-1}B_{n+1}^{[1]}(x)$, $n \geq 0$, satisfies

$$\langle u_n^{[1]} \rangle' = -(n + 1)u_{n+1}, \quad n \geq 0. \quad (1.1)$$

Definition 1.2. The linear functional $u$ is said to be regular if there exists a monic polynomial sequence $\{B_n\}_{n \geq 0}$ such that

$$\langle u, B_n B_m \rangle = b_n \delta_{n,m}, \quad n, m \geq 0. \quad (1.2)$$

where

$$b_n = \langle u, B_n^2 \rangle \neq 0, \quad n \geq 0. \quad (1.3)$$

Then the sequence $\{B_n\}_{n \geq 0}$ is said to be orthogonal (MOPS) with respect to $u$.

As a straightforward consequence we get

- The linear functional can be represented by $u = (u_0)_{u_0}$, and the following relations hold

$$u_n = b_n^{-1}B_n u, \quad n \geq 0. \quad (1.4)$$
The sequence \( \{B_n\}_{n \geq 0} \) satisfies the three-term recurrence relation
\[
B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0,
\]
\[
B_0(x) = x - \beta_0, \quad B_0(x) = 1,
\]
where \( \gamma_{n+1} \neq 0, \quad n \geq 0 \) (see [4]).

In the sequel and under the assumption of the previous definition, we need to put
\[
b_{n,m}^\nu = b_m^{-1}(u, x^\nu B_mB_n), \quad (n, \nu, m) \in \mathbb{N}^3.
\]
In particular, one has
\[
b_{n,m}^\nu = \begin{cases} 0, & \text{if } \nu + m < n, \quad 0 \leq m < n, \quad \nu \geq 0, \\
(b_n/b_m), & \text{if } \nu = n - m, \quad 0 \leq m \leq n.
\end{cases}
\]

Let \( \phi \) be a monic polynomial, with \( \deg \phi = t \geq 0 \). For any MPS \( \{B_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) with dual sequences \( \{u_n\}_{n \geq 0} \) and \( \{v_n\}_{n \geq 0} \) respectively, the following formula always holds
\[
\phi(x)Q_n(x) = \sum_{\nu=0}^{n+t} \lambda_{n,\nu} B_{\nu}(x), \quad n \geq 0,
\]
where \( \lambda_{n,\nu} = \langle u_{\nu}, \phi Q_n \rangle, \quad 0 \leq \nu \leq n + t, \quad n \geq 0. \)

**Definition 1.3.** ([12]) If there exists an integer \( s \geq 0 \) such that
\[
\phi(x)Q_n(x) = \sum_{\nu=0}^{n+t} \lambda_{n,\nu} B_{\nu}(x), \quad n \geq s,
\]
and
\[
\exists r \geq s, \lambda_{r,r-s} \neq 0,
\]
then, we shall say that (1.8) – (1.9) gives a finite-type relation between \( \{B_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \), with respect to \( \phi \).

When instead of (1.9), we take
\[
\lambda_{n,n-s} \neq 0, \quad n \geq s,
\]
we shall say that (1.8) – (1.9)' is a strictly finite-type relation.

The corresponding inverse finite-type relation of (1.8) – (1.9) consists in establishing, whenever it is possible, a finite-type relation between \( \{Q_n\}_{n \geq 0} \) and \( \{B_n\}_{n \geq 0} \), as follows
\[
\Omega_s(x; n)B_n(x) = \sum_{\nu=n-t}^{n+s} \theta_{n,\nu}^* Q_{\nu}(x), \quad n \geq t,
\]
where \( \{\Omega_s(x; n)\}_{n \geq t} \) is a MPS, \( \deg \Omega_s(x; n) = s, \quad n \geq t, \) and
\[
(\theta_{n,\nu}^*)_{\nu=n-t}^{n+s}, \quad n \geq t,
\]
a system of complex numbers (SCN), with \( \theta_{n,n+s}^* = 1, \quad n \geq t. \)

When both two sequences are orthogonal, the inverse relation is always possible. In this case, the polynomials \( \Omega_s(x; n), \quad n \geq 0, \) are independent of \( n \), (see [12], Proposition 2.4). As a current example, we can mention the two structure relations characterizing the classical polynomials, (Hermite, Laguerre, Bessel, Jacobi, see [11]), which could solely be two inverse finite-type relations.

In other studies, we find several situations where one of the two sequences is orthogonal. For example, the structure relations characterizing semi-classical sequences associated with Hahn’s operators \( L_{q,\omega} \), with parameters \( q \) and \( \omega \), [9]. The Coherent pairs and Diagonal sequences are also examples of finite-type relations [7, 12, 13, 14]. But the inverse relations corresponding to other finite-type relations are not yet considered.

The paper essentially gives a necessary and sufficient condition allowing the existence of the inverse finite-type relations when the orthogonality of the sequence \( \{B_n\}_{n \geq 0} \) is assumed. From now on, it would be necessary to study the case where the sequence \( \{Q_n\}_{n \geq 0} \) is orthogonal. It would be very useful to deal with many other situations like General Coherent pairs, see [0, 8] in the framework of Sobolev inner products.

### 2. A basic result

We use this section to introduce some auxiliary result for the proof of the main theorem in section 3.

**Lemma 2.1.** Suppose \( \{B_n\}_{n \geq 0} \) is a MOPS and \( \{Q_n\}_{n \geq 0} \) fulfills (1.8) – (1.9), where \( t = 0 \) and \( s \geq 1 \). For any SCN \( (\theta_{n,\nu}^*)_{\nu=n-s}^{n+s}, \quad n \geq 0, \) where \( \theta_{n,n+s} = 1, \quad n \geq 0, \) and \( \theta_{r,r} \neq 0, \) there exist a unique MPS \( \{\Omega_s(x; n)\}_{n \geq 0}, \) \( \deg \Omega_s(x; n) = s, \quad n \geq 0, \) and a SCN \( (\zeta_{n,\nu}^*)_{\nu=n-s}^{n-1}, \quad n \geq 0, \) such that
\[
\sum_{\nu=n}^{n+s} \theta_{n,\nu} Q_{\nu}(x) = \sum_{i=n-s}^{n+s} \theta_{n,s}^{[0]} B_i(x)
\]
\[
= \Omega_s(x; n)B_n(x) + \sum_{\nu=n-s}^{n-1} \zeta_{n,\nu}^* B_{\nu}(x), \quad n \geq 0.
\]
where
\[ \theta_{n,i}^{[0]} = \sum_{\nu = \max(n,i)}^{\min(n,i)+s} \theta_{n,\nu} \lambda_{\nu, i}, \quad n-s \leq i \leq n+s, n \geq 0, \] (2.2)

\[ \theta_{r,r-s}^{[0]} = \theta_{r,r} \lambda_{r, r-s} \neq 0, \] (2.3)

\[ \sum_{\nu = n}^{m+s} \theta_{n,\nu} \lambda_{\nu, m} = b_{m}^{-1}(u, \Omega_{s}(x; n) B_{n} B_{m}) + \zeta_{n,m}^{[0]}, \quad n-s \leq m \leq n-1, n \geq 0, \] (2.4)

\[ \sum_{\nu = m}^{n+s} \theta_{n,\nu} \lambda_{\nu, m} = b_{m}^{-1}(u, \Omega_{s}(x; n) B_{n} B_{m}), \quad n \leq m \leq n+s-1, n \geq 0. \] (2.5)

Proof. Let \((\theta_{n,\nu})_{n \geq 0}, n \geq 0\), where \(\theta_{n,n+s} = 1, n \geq 0\), and \(\theta_{r,r} \neq 0\), be a SCN. From (1.8) – (1.9), with \(t = 0\) and \(s \geq 1\), we get

\[ \sum_{\nu = n}^{n+s} \theta_{n,\nu} Q_{\nu}(x) = \sum_{\nu = n}^{n+s} \theta_{n,\nu} \sum_{i = \nu-s}^{\nu} \lambda_{\nu, i} B_{i}(x) = \sum_{\nu = n}^{n+s} \theta_{n,\nu} \sum_{i = \nu-s}^{\nu} \chi_{\nu, \nu} \lambda_{\nu, i} B_{i}(x), n \geq 0, \]

where, for each pair of integers \((i, \nu)\) such that \(n-s \leq i \leq n+s\) and \(n \leq \nu \leq n+s\), we took

\[ \chi_{\nu, \nu} = \begin{cases} 1, & \text{if } \nu - s \leq i \leq \nu, \\ 0, & \text{otherwise.} \end{cases} \]

The permutation of these two sums yields

\[ \sum_{\nu = n}^{n+s} \theta_{n,\nu} Q_{\nu}(x) = \sum_{i = n-s}^{n+s} \theta_{n,i}^{[0]} B_{i}(x), \]

where

\[ \theta_{n,i}^{[0]} = \sum_{\nu = \max(n,i)}^{\min(n,i)+s} \theta_{n,\nu} \lambda_{\nu, i}, \quad n-s \leq i \leq n+s, n \geq 0, \] (2.6)

\[ \theta_{r,r-s}^{[0]} = \theta_{r,r} \lambda_{r, r-s} \neq 0. \] (2.7)

Hence, (2.2) and (2.3) are valid.

The Euclidean division by \(B_{n}(x)\) in the right hand side in (2.6) gives

\[ \sum_{i = n-s}^{n+s} \theta_{n,i}^{[0]} B_{i}(x) = \Omega_{s}(x; n) B_{n}(x) + \sum_{\nu = 0}^{n-1} \zeta_{n,\nu}^{[0]} B_{\nu}(x), n \geq 0. \]

Multiplying by \(B_{m}(x)\) and using the orthogonality of \(\{B_{m}\}_{n \geq 0}\),

\[ \sum_{i = n-s}^{n+s} \theta_{n,i}^{[0]} \delta_{m,i} = b_{m}^{-1}(u, \Omega_{s}(x; n) B_{n} B_{m}) + \sum_{\nu = 0}^{n-1} \zeta_{n,\nu}^{[0]} \delta_{m,\nu}. \]

In particular, for \(0 \leq m \leq n-s-1\) and \(n \geq s+1\), it follows that \(\zeta_{n,m}^{[0]} = 0\). Hence, (2.1) holds. Moreover, for \(n-s \leq m \leq n-1\) and \(n \geq s\), we recover (2.4).

Finally, for \(n \leq m \leq n+s-1\) and \(n \geq 0\), we deduce (2.5).

**Proposition 2.2.** Assume \(\{B_{n}\}_{n \geq 0}\) is a MOPS and \(\{Q_{n}\}_{n \geq 0}\) fulfills (1.8) – (1.9), with \(t \geq 1\). For any SCN \((\theta_{n,\nu})_{n \geq 0}, n \geq 0\), where \(\theta_{n,n+s} = 1, n \geq 0\) and \(\theta_{r,r} \neq 0\), there exist a unique MPS \(\{\Omega_{s+t}(x; n)\}_{n \geq 0}\), where \(\deg \Omega_{s+t}(x; n) = s+t, n \geq 0\), and a SCN \((\zeta_{n,\nu})_{n \geq 0}

\[ \phi(x) \sum_{\nu = n-s-t}^{n+s+t} \theta_{n,\nu} Q_{\nu}(x) = \sum_{i = n-s-t}^{n+s+t} \theta_{n,i}^{[t]} B_{i}(x) = \Omega_{s+t}(x; n) B_{n}(x) + \sum_{\nu = n-s-t}^{n-1} \zeta_{n,\nu}^{[t]} B_{\nu}(x), \] (2.7)

where

\[ \theta_{n,i}^{[t]} = \sum_{\nu = \max(n,i)}^{\min(n,i)+s+t} \theta_{n,\nu} \lambda_{\nu, i}, \quad n-s-t \leq i \leq n+s+t, \] (2.8)

\[ \theta_{r+r,t-r}^{[t]} = \theta_{r+r} \lambda_{r, r-t} \neq 0, \] (2.9)

\[ \sum_{\nu = n}^{n+s+t} \theta_{n,\nu} \lambda_{\nu, t-m} = b_{m}^{-1}(u, \Omega_{s+t}(x; n) B_{n} B_{m}) + \zeta_{n,m}^{[t]}, \quad n-s-t \leq m \leq n-1, \] (2.10)

\[ \sum_{\nu = m}^{n+s+t} \theta_{n,\nu} \lambda_{\nu, t-m} = b_{m}^{-1}(u, \Omega_{s+t}(x; n) B_{n} B_{m}), \quad n \leq m \leq n+s+t-1. \] (2.11)

Proof. The case \(t = 0\) was analyzed in Lemma 2.1. Let us take \(t \geq 1\). Consider the MPS \(\{P_{n}\}_{n \geq t}\) defined by

\[ P_{n+t}(x) = \phi(x) Q_{n}(x), n \geq 0, \] (2.12)
From (1.8) – (1.9), we have

\[ P_n(x) = \sum_{\nu=n-t-s}^{n} \tilde{\lambda}_{n,\nu} B_{\nu}(x), \quad n \geq t, \]

where \( \tilde{\lambda}_{n,\nu} = \lambda_{n-t,\nu}, \quad n - t - s \leq \nu \leq n, \quad n \geq t, \) and \( \tilde{\lambda}_{n+t,r-s} \neq 0. \) Now, let \( (\theta_{n,\nu})_{\nu=n-t}^{n+t}, \quad n \geq 0, \) where \( \theta_{n,n+s} = 1, \quad n \geq 0, \) and \( \theta_{r+t,r} \neq 0, \) be a SCN. One has

\[ \phi(x) \sum_{\nu=n-t}^{n+s} \theta_{n,\nu} Q_{\nu}(x) = \sum_{\nu=n-t}^{n+s} \theta_{n,\nu} P_{\nu+t}(x) \]
\[ = \sum_{\nu=n}^{n+t+s} \tilde{\theta}_{n,\nu} P_{\nu}(x), \quad n \geq 0, \quad (2.13) \]

where \( \tilde{\theta}_{n,\nu} = \theta_{n,\nu-t}, \quad n \leq \nu \leq n + t + s, \quad n \geq 0. \) Obviously, \( (\tilde{\theta}_{n,\nu})_{\nu=n-t}^{n+s+t}, \quad n \geq 0, \) is a SCN such that

\[ \tilde{\theta}_{n,n+s+t} = \theta_{n,n+s} = 1, \quad n \geq 0, \quad \tilde{\theta}_{r+t,r+t} = \theta_{r+t,r} \neq 0. \]

But from Lemma 2.1, there exist a unique MPS \( \{\Omega_{n+s}(x; n)\}_{n \geq 0} \) and a SCN \( (\zeta_{n,\nu})_{\nu=n-t-s}^{n-1}, \quad n \geq 0, \) such that

\[ \sum_{\nu=n-t-s}^{n+t+s} \tilde{\theta}_{n,\nu} P_{\nu}(x) = \sum_{i=n-t-s}^{n+t+s} \theta_{n,i} B_{i}(x) \]
\[ = \Omega_{n+s}(x; n) B_{n}(x) + \sum_{\nu=n-t-s}^{n-1} \zeta_{n,\nu} B_{\nu}(x), \quad (2.14) \]

for every integer \( n \geq 0, \) where

\[ \theta_{n,i}^{[t]} = \frac{\min(n,i)+t+s}{\max(n,i)}, \quad \theta_{r+t,r+s}^{[t]} = \tilde{\theta}_{r+t,r+t+s} \neq 0, \]

\[ \sum_{\nu=n-t-s}^{n+t+s} \tilde{\theta}_{n,\nu} \tilde{\lambda}_{n,\nu}(x) = b_{m}^{-1}(u, \Omega_{n+s}(x; n) B_{n} B_{m}) + \zeta_{n,m}, \quad n - t - s \leq m \leq n - 1, \]

\[ \sum_{\nu=m}^{n+t+s} \tilde{\theta}_{n,\nu} \tilde{\lambda}_{n,\nu}(x) = b_{m}^{-1}(u, \Omega_{n+s}(x; n) B_{n} B_{m}), \quad n \leq m \leq n + t + s - 1. \]

Finally, by using (2.13), (2.14), and taking into account the expressions of \( \tilde{\lambda}_{n,\nu} \) and \( \tilde{\theta}_{n,\nu}, \) we find the desired results.

3. A matrix approach and main results

In this section, we will work under the assumptions of the Proposition 2.2 and we will give a matrix approach to our problem.

If \( \omega_{n,t+s}(x; n) = \sum_{\nu=0}^{t+s} v_{n,\nu} x^{\nu}, \quad n \geq 0, \) where \( v_{n,t+s} = 1, \) then relation (2.10) reads

\[ \sum_{\nu=n}^{m+s+t} \lambda_{n-t,m} \theta_{n,\nu-t} = \sum_{\nu=0}^{t+s-1} b_{n,m}^{\nu} v_{n,\nu} + c_{n,m}^{[t]} + b_{n,m}^{s+t}, \quad n - s - t \leq m \leq n - 1, \]

or, alternatively,

\[ \sum_{j=1}^{m+s+t-n-1} \lambda_{j+n-t-1,m} \theta_{n,j+n-t-1} = \sum_{j=1}^{t+s} b_{n,m}^{j-1} v_{n,j-1} + c_{n,m}^{[t]} + b_{n,m}^{s+t}, \]

for every \( n - s - t \leq m \leq n - 1. \)

Replacing \( m \) by \( i + n - s - t - 1, \) we get

\[ \sum_{j=1}^{t+s} t_{n,j}^{[i]} \Theta_{n,j} = \sum_{j=1}^{t+s} t_{n,j}^{[i]} V_{n,j} + c_{n,i+n-s-t-1} + b_{n,i+n-s-t-1}^{s+t}, \quad 1 \leq i \leq s + t, \]

where for \( i, j = 1, 2, \ldots, s + t, \)

\[ k_{i,j}^{[n]} = \begin{cases} \lambda_{j+n-t-1,i+n-s-t-1}, & 1 \leq j \leq i \\ 0, & \text{otherwise}, \end{cases} \]

and \( t_{n,j}^{[i]} = b_{n,i+n-s-t-1}^{j-1}, \)

\( \Theta_{n,j} = \theta_{n,t+n-1}, \) and \( V_{n,j} = v_{n,j-1}. \)

So we can write it as

\[ K_{n} \Theta_{n} = T_{n} V_{n} + W_{n} + E_{n}, \quad n \geq 0, (3.1) \]

where

\[ K_{n} = \left( k_{i,j}^{[n]} \right)_{1 \leq i,j \leq s+t}, \quad T_{n} = \left( t_{i,j}^{[n]} \right)_{1 \leq i,j \leq s+t}, \]

\[ \Theta_{n} = \left( \Theta_{n,1}, \Theta_{n,2}, \ldots, \Theta_{n,s+t} \right)^{T}, \]

\[ V_{n} = \left( V_{n,1}, V_{n,2}, \ldots, V_{n,s+t} \right)^{T}, \]

\[ W_{n} = \left( c_{n,s-t}^{[t]}, c_{n,n+1-s-t}^{[t]}, \ldots, c_{n,n-1}^{[t]} \right)^{T}, \] and

\[ E_{n} = \left( b_{n,n-s-t}^{s+t}, b_{n,n+1-s-t}^{s+t}, \ldots, b_{n,n-1}^{s+t} \right)^{T}. \]
In the same way, using $\theta_{n,n+s} = 1$, (2.11) can be written as
\[
\sum_{\nu=m}^{n+s+t-1} \lambda_{\nu-t,m} \theta_{\nu,n} = \sum_{\nu=0}^{t+s-1} b_{n,m}^{\nu} \nu_{\nu} + b_{n,m}^{n+s,t} - \lambda_{n+s,m},
\]
for every $n \leq m \leq n+s+t-1$. Replacing $m$ by $i+n-1$, we get
\[
\sum_{j=1}^{t+s} m_{i,j}^{n} \theta_{i+j} = \sum_{j=1}^{t+s} s_{i,j}^{n} \nu_{i+j} + b_{n,i+n-1}^{t+s} - \lambda_{n+s,i+n-1},
\]
where for $i, j = 1, 2, ..., s + t$,
\[
m_{i,j}^{n} = \begin{cases} 
\lambda_{j+n-1,i+n-1}, & 1 \leq i \leq j \\
0, & \text{otherwise}, 
\end{cases}
\]
and
\[
s_{i,j}^{n} = \begin{cases} 
b_{n,i+n-1}, & 1 \leq i \leq j \\
0, & \text{otherwise}. 
\end{cases}
\]
Thus, we can use the matrix representation
\[
M_{n} \theta_{n} = S_{n} V_{n} + F_{n},
\]
where
\[
M_{n} = (m_{i,j}^{n})_{1 \leq i, j \leq s + t}, \quad S_{n} = (s_{i,j}^{n})_{1 \leq i, j \leq s + t},
\]
and
\[
F_{n} = (b_{n,n}^{t+s} - \lambda_{n+s,n}, b_{n,n+1}^{t+s} - \lambda_{n+s,n+1}, \ldots, b_{n,n+s+t-1}^{t+s} - \lambda_{n+s,n+s+t-1}).
\]
Our data are $\theta_{n}$, $E_{n}$, $F_{n}$, $M_{n}$, $S_{n}$, $T_{n}$, $K_{n}$ and our unknowns are $V_{n}$ and $W_{n}$.

From (3.2), we get
\[
V_{n} = S_{n}^{-1}(M_{n} \theta_{n} - F_{n}).
\]

Thus, substituting in (3.1) we get $K_{n} \theta_{n} - W_{n} - E_{n} = T_{n} S_{n}^{-1}(M_{n} \theta_{n} - F_{n})$, i.e.,
\[
W_{n} = (K_{n} - T_{n} S_{n}^{-1} M_{n}) \theta_{n} + T_{n} S_{n}^{-1} F_{n} - E_{n}.
\]
As a consequence, for every choice of $\theta_{n}$, we get $W_{n}$. From (3.3), we deduce $V_{n}$.

On the other hand, there exists a one-to-one correspondence between the vectors $W_{n}$ and $\Theta_{n}$ if and only if the matrix of dimension $s + t$, $K_{n} - T_{n} S_{n}^{-1} M_{n}$, is nonsingular.

Under such a condition, there exists a unique choice for $\Theta_{n}$ such that $W_{n} = 0$. Thus, we get
\[
\Theta_{n} = (K_{n} - T_{n} S_{n}^{-1} M_{n})^{-1}(E_{n} - T_{n} S_{n}^{-1} F_{n}),
\]
and from (3.3), $V_{n} = S_{n}^{-1} M_{n} \theta_{n} - S_{n}^{-1} F_{n}$.
Then,
\[
V_{n} = (K_{n} M_{n}^{-1} S_{n} - T_{n})^{-1} E_{n} - [(K_{n} M_{n}^{-1} S_{n} - T_{n})^{-1} T_{n} + I_{s+t}] S_{n}^{-1} F_{n},
\]
where $I_{s+t}$ is the unit matrix. Hence, the polynomial $\Omega_{s+t}(x;n)$ is explicitly given.

Let introduce
\[
\Delta_{n}(t,s) = \det(K_{n} - T_{n} S_{n}^{-1} M_{n}), n \geq 0.
\]
Thus, we have proved the following result

**Proposition 3.1.** Assume $\{B_{n}\}_{n \geq 0}$ is a MOPS and $\{Q_{n}\}_{n \geq 0}$ fulfills (1.8) – (1.9). For a fixed integer $p \geq t + 1$, the following statements are equivalent.

i) $\Delta_{n}(t,s) \neq 0$, $n \geq p$.

ii) There exist a unique SCI $(\theta^{*}_{n,v})_{v=0}^{n+s}$, $n \geq p$, with $\theta^{*}_{n,n+s} = 1$, $n \geq p$, and $\theta^{*}_{r+t,r} \neq 0$, if $p \leq r + t$, and a unique MPS $\{\Omega^{*}_{s+t}(x;n)\}_{n \geq p}$, deg $\Omega^{*}_{s+t}(x;n) = s + t$, $n \geq p$, such that
\[
\Omega^{*}_{s+t}(x;n) B_{n}(x) = \phi(x) \sum_{\nu=n-t}^{n+s} \theta^{*}_{n,v} Q_{\nu}(x),
\]
for $n \geq p$.

Our main result is

**Theorem 3.2.** Let $\{B_{n}\}_{n \geq 0}$ be a MOPS and $\{Q_{n}\}_{n \geq 0}$ be the MPS satisfying (1.8) – (1.9). For each fixed integer $p \geq t + 1$, if we suppose that $\phi(x)$ and $B_{n}(x)$ are coprime for every $n \geq p$, then the following statements are equivalent.

i) $\Delta_{n}(t,s) \neq 0$, $n \geq p$.

ii) There exist a unique SCI $(\theta^{*}_{n,v})_{v=0}^{n+s}$, $n \geq p$, where $\theta^{*}_{n,n+s} = 1$, $n \geq p$, and $\theta^{*}_{r+t,r} \neq 0$ if $p \leq r + t$, and a unique MPS $\{\Omega^{*}_{s+t}(x;n)\}_{n \geq p}$, deg $\Omega^{*}_{s+t}(x;n) = s$, $n \geq p$, such that
\[
\Omega^{*}_{s+t}(x;n) B_{n}(x) = \sum_{\nu=n-t}^{n+s} \theta^{*}_{n,v} Q_{\nu}(x), n \geq p.
\]
Proof. Taking into account \( \phi(x) \) and \( B_n(x) \) are coprime for every \( n \geq p \), from (3.4) we deduce that \( \phi \) divides \( \Omega_{s+i}(x; n) \), \( n \geq p \). So, \( \Omega_{s+i}(x; n) = \phi(x)\Omega_{s}(x; n) \), \( n \geq p \). Hence, the desired result follows.

The orthogonal polynomial sequence \( \{B_n\}_{n \geq 0} \) and the polynomial sequence \( \{Q_n\}_{n \geq 0} \) can be related by a general finite-type relation (see [1]). It reads as follows

\[
F(Q_n, \ldots, Q_{n-l}) = G(B_n, \ldots, B_{n-s}),
\]

where \( F \) and \( G \) are fixed functions.

When \( F \) and \( G \) are linear functions, some situations dealing with the inverse problem have been analyzed in [1,2]. There, necessary and sufficient conditions in order to \( \{Q_n\}_{n \geq 0} \) be orthogonal are obtained.

This kind of linear relations reads as follows.

There exists \( (l, s, r) \in \mathbb{N}^3 \), with \( r \geq s = \max(l, s) \) such that

\[
\sum_{\nu=n-l}^{n} \zeta_{n,\nu}Q_{\nu}(x) = \sum_{\nu=n-s}^{n} \lambda_{n,\nu}B_{\nu}(x), \quad n \geq s,
\]

with \( \zeta_{r, r-l}\lambda_{r, r-s} \neq 0 \). Here, \( \zeta_{n,n} = \lambda_{n,n} = 1, \quad n \geq s \).

More recently, in [5], A. M. Delgado and F. Marcellán exhaustively describe all the set of pairs of quasi-definite (regular) linear functionals such that their corresponding sequences of monic polynomials \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) are related by a differential expression

\[
P_n(x) + s_nP_{n-1}(x) = R_n^{[1]}(x) + t_nR_{n-1}^{[1]}(x), \quad n \geq 1,
\]

where \( t_n \neq 0 \), for every \( n \geq 1 \), and with the technical condition \( t_1 \neq s_1 \).

Notice that in general \( \{R_n^{[1]}\}_{n \geq 0} \) is not a MOPS.

In the same context of our contribution, we show that the corresponding inverse finite-type relation between two sequences satisfying (3.6) is possible under certain conditions.

Indeed, let consider the MPS \( \{C_n\}_{n \geq s} \) given by

\[
C_n(x) = \sum_{\nu=n-s}^{n} \lambda_{n,\nu}B_{\nu}(x), \quad n \geq s.
\]

With the finite-type relation between the sequences \( \{C_n\}_{n \geq s} \) and \( \{B_n\}_{n \geq s} \), we can associate the determinants \( \Delta_n(0, s) \), \( n \geq s \). So, we have.

Corollary 3.3. Let \( \{B_n\}_{n \geq 0} \) be a MOPS and \( \{Q_n\}_{n \geq 0} \) be the MPS satisfying (3.6). For each fixed integer

\[
p \geq \max(s, l, 1), \quad \text{if } \Delta_n(0, s) \neq 0, \quad n \geq p, \text{ then there exist a unique SCN } \left( \zeta_{n,\nu}^{*}\right)_{\nu=n-l}^{n+s-1}, \quad n \geq p, \text{ where } \zeta_{n,n+s}^{*} = 1, \quad n \geq p, \text{ and } \zeta_{r,r-l}^{*} \neq 0 \text{ if } p \leq r, \text{ and a unique MPS } \{\Omega_{s}^{*}(x; n)\}_{n \geq p}, \deg \Omega_{s}^{*}(x; n) = s, \quad n \geq p, \text{ such that}
\]

\[
\Omega_{s}^{*}(x; n)B_n(x) = \sum_{\nu=n-l}^{n+s} \zeta_{n,\nu}^{*}Q_{\nu}(x), \quad n \geq p.
\]

Proof. From Theorem 3.2, with \( t = 0 \), there exists the corresponding inverse finite-type relation associated with the relation (3.7) if and only if \( \Delta_n(0, s) \neq 0, \quad n \geq p \). Equivalently, there exist a unique SCN \( \left( \theta_{n,\nu}^{*}\right)_{\nu=n-l}^{n+s-1}, \quad n \geq p, \text{ where } \theta_{n,n+s}^{*} = 1, \quad n \geq p, \text{ and } \theta_{r,r-l}^{*} \neq 0, \text{ if } p \leq r, \text{ and a unique MPS } \{\Omega_{s}^{*}(x; n)\}_{n \geq p}, \deg \Omega_{s}^{*}(x; n) = s, \quad n \geq p, \text{ such that}
\]

\[
\Omega_{s}^{*}(x; n)B_n(x) = \sum_{\nu=n-l}^{n+s} \theta_{n,\nu}^{*}C_{\nu}(x), \quad n \geq p.
\]

But from (3.6) and (3.7), the above expression becomes

\[
\Omega_{s}^{*}(x; n)B_n(x) = \sum_{\nu=n}^{n+s} \theta_{n,\nu}^{*} \sum_{i=\nu-l}^{\nu} \zeta_{\nu,i}Q_{i}(x)
\]

\[
= \sum_{\nu=n}^{n+s} \theta_{n,\nu}^{*} \sum_{i=\nu-l}^{\nu} \chi_{\nu,i}Q_{i}(x), \quad n \geq p,
\]

where, for each pair of integers \( (i, \nu) \) such that \( n - l \leq i \leq n + s \) and \( n \leq \nu \leq n + s \), we took

\[
\chi_{\nu,i} = \begin{cases} 1, & \text{if } \nu - l \leq i \leq \nu, \\ 0, & \text{otherwise.} \end{cases}
\]

The permutation inside these two sums yields

\[
\Omega_{s}^{*}(x; n)B_n(x) = \sum_{i=n-l}^{n+s} \zeta_{n,\nu}^{*}Q_{i}(x),
\]

where

\[
\zeta_{n,\nu}^{*} = \sum_{\nu=\max(n, i)}^{\min(n+s, n+l)} \theta_{n,\nu}^{*}\zeta_{\nu,i},
\]

if \( n - l \leq i \leq n + s, \quad n \geq p, \text{ and }
\]

\[
\zeta_{r,r-l}^{*} = \theta_{r,r-l}^{*} \chi_{r,r-l}^{*} \neq 0, \quad \text{if } p \leq r.
\]
4. The case: $(t, s) = (0, 1)$

Let $\{B_n\}_{n \geq 0}$ be a MOPS with respect to the linear functional $u$ and satisfying the three-term recurrence relation (1.5).

Consider the following finite-type relation between $\{B_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, with index $s = 1$, with respect to $\phi(x) = 1$,

$$Q_n(x) = B_n(x) + \lambda_{n,n-1} B_{n-1}(x), \quad n \geq 0, \quad (4.1)$$

$$\forall r \geq 1, \quad \lambda_{r,r-1} \neq 0. \quad (4.2)$$

From Lemma 2.1, for every set of complex numbers, $\theta_{n,n}$, $n \geq 0$, with $\theta_{r,r} \neq 0$, there exists a unique MPS $\{\Omega_1(x;n)\}_{n \geq 0}$, where $\Omega_1(x;n) = x + v_{n,0}$, $n \geq 0$, and a unique set of complex numbers, $\zeta_{n,n-1}$, $n \geq 0$, such that

$$Q_{n+1}(x) + \theta_{n,n} Q_n(x) = \Omega_1(x;n) B_n(x) + \zeta_{n,n-1} B_{n-1}(x), \quad n \geq 0, \quad (4.3)$$

where

$$\left\{ \begin{array}{ll}
\lambda_{n,n-1} \theta_{n,n} = \zeta_{n,n-1} + \gamma_n, & n \geq 1, \\
\theta_{n,n} - v_{n,0} = -\lambda_{n+1,n} + \beta_n, & n \geq 0.
\end{array} \right. \quad (4.4)$$

The determinants associated with (4.1)–(4.2) are given by

$$\Delta_0(0, 1) = 0, \quad \Delta_n(0, 1) = \lambda_{n,n-1}, \quad n \geq 1, \quad (4.5)$$

where $\Delta_r(0, 1) = \lambda_{r,r-1} \neq 0$. As a consequence of Theorem 3.2, when $t = 0$ and $s = 1$, we have the following result

**Proposition 4.1.** Let $\{B_n\}_{n \geq 0}$ be a MOPS and $\{Q_n\}_{n \geq 0}$ be the MPS satisfying (4.1)–(4.2). For every fixed integer $p \geq 1$, the following statements are equivalent

i) $\lambda_{n,n-1} \neq 0$, $n \geq p$.

ii) There exist a unique set of complex numbers $\theta_{n,n}$, $n \geq p$, and a unique MPS $\{\Omega_1(x;n)\}_{n \geq p}$, deg $\Omega_1(x;n) = 1$, $n \geq p$, such that

$$\Omega_1^*(x;n) B_n(x) = Q_{n+1}(x) + \theta_{n,n}^* Q_n(x), \quad n \geq p. \quad (4.6)$$

We write

$$\theta_{n,n}^* = \frac{\gamma_n}{\lambda_{n,n-1}}, \quad n \geq p, \quad (4.7)$$

$$\Omega_1^*(x;n) = x + v_{n,0}^*,$$

where

$$v_{n,0}^* = \frac{\gamma_n}{\lambda_{n,n-1}} + \lambda_{n+1,n} - \beta_n, \quad n \geq p. \quad (4.8)$$

**Example.** In order to illustrate the result of Proposition 4.1, we study the structure relation characterizing a semi-classical polynomial sequence, $\{B_n\}_{n \geq 0}$, orthogonal with respect to the linear functional $u$ solution of the functional equation

$$u' + \psi u = 0, \quad (4.9)$$

where $\psi(x) = -ix^2 + 2x - i(\alpha - 1)$ and with regularity condition $\alpha \notin \bigcup_{n \geq 0} E_n$, where $E_0 = \{\alpha \in \mathbb{C} : F(\alpha) = 0\}$, $F(\alpha) = \int_{-\infty}^{+\infty} e^{i\alpha x} x^{2+\epsilon(\alpha-1)} dx$, and for each integer $n \geq 1$, $E_n = \{\alpha \in \mathbb{C} : \Xi_n(\alpha) = 0\}$. Here, $\Xi_n(\alpha)$ is the Hankel determinant associated with $u$. Notice that $u$ is a semi-classical linear functional of class one $[10]$.

The recurrence coefficients $\beta_n$ and $\gamma_{n+1}$, $n \geq 0$, of the sequence $\{B_n\}_{n \geq 0}$ are determined by the system $[10]$

$$\begin{bmatrix}
\frac{n+1}{\gamma_{n+1}} \\
\frac{i(\gamma_{n+2} + \gamma_{n+1})}{\gamma_1} \\
\frac{i(\gamma_{n+2})}{\gamma_1}
\end{bmatrix}
= 2 - i(\beta_n + \beta_{n+1}), \quad n \geq 0, \quad (4.10)$$

$$\Gamma_1 = -i\psi(\beta_0), \quad \beta_0 = -\frac{i}{F'(\alpha)}.$$

The sequence $\{B_n\}_{n \geq 0}$ is characterized by the following structure relation $[10]$

$$B_n^{[1]}(x) = B_n(x) - \frac{i\gamma_n}{n+1} B_{n+1}(x), \quad n \geq 1. \quad (4.11)$$

Thus, taking into account $\lambda_{n,n-1} = -\frac{i\gamma_n}{n+1} \neq 0, \quad n \geq 1$, we deduce a strictly finite-type relation between the sequences $\{B_n\}_{n \geq 0}$ and $\{B_n^{[1]}\}_{n \geq 0}$ with index $s = 1$, with respect to $\phi(x) = 1$,

From Proposition 4.1, we get the following inverse relation, for $n \geq 1$,

$$(x + v_{n,0}^*) B_n(x) = B_{n+1}(x) + \frac{i(n+1)}{\gamma_{n+1}} B_n^{[1]}(x), \quad (4.12)$$

where $v_{n,0}^* = \frac{i(n+1)}{\gamma_{n+1}} - \frac{i\gamma_{n+1}\gamma_{n+2}}{n+2} - \beta_n, \quad n \geq 1$. The sequence $\{B_n\}_{n \geq 0}$ could be characterized by a relation as (4.12). It is the aim of the following result.

**Proposition 4.2.** Let $\{B_n\}_{n \geq 0}$ be a MOPS satisfying (1.5). Then the following statements are equivalent.

i) There exists a set of non-zero complex numbers $\{\lambda_{n,n-1}\}_{n \geq 1}$ such that, for $n \geq 1$,

$$B_n^{[1]}(x) = B_n(x) + \lambda_{n,n-1} B_{n-1}(x). \quad (4.13)$$
ii) There exists a set of complex numbers \( \{ \varrho_n \}_{n \geq 0} \), with \( \varrho_n \neq 0 \), \( n \geq 1 \), and \( \varrho_0 = 0 \), such that for \( n \geq 0 \),

\[
(x + \frac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n)B_n(x) = B_{n+1}^{[1]}(x) + \varrho_nB_n^{[1]}(x),
\]

(4.14)

Proof. Assume that i) holds. From Proposition 4.1, we get

\[
(x + \frac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n)B_n(x) = B_{n+1}^{[1]}(x) + \varrho_nB_n^{[1]}(x), \quad n \geq 1,
\]

where \( \varrho_n = \gamma_n \lambda_{n-1}^{-1}, \ n \geq 1 \). For \( n = 1 \), in (4.13), we obtain \( \lambda_{1,0} = \frac{\beta_0 - \beta_1}{2} \). Then, \( \frac{\gamma_1}{\varrho_1} = \frac{\beta_0 - \beta_1}{2} \). Hence,

\[
(x + \frac{\gamma_1}{\varrho_1} - \beta_0)B_0(x) = x - \frac{\beta_0 + \beta_1}{2} = B_1^{[1]}(x) + \varrho_0B_0^{[1]}(x),
\]

i.e. \( \varrho_0 = 0 \). Thus, ii) holds. Conversely, let us take \( \lambda_{n-1} = \frac{\gamma_n}{\varrho_n}, \ n \geq 1 \), and consider the MPS \( \{ A_n \}_{n \geq 0} \) defined by

\[
A_n(x) = B_n(x) + \lambda_{n-1}B_{n-1}(x), \quad n \geq 1.
\]

(4.15)

From Proposition 4.1, we get

\[
(x + \varrho_{n,0}^* A_n(x) = A_{n+1}(x) + \varrho_{n+1}^* A_n(x), \quad n \geq 1,
\]

where \( \varrho_{n,0} = \frac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n, \ n \geq 1 \), and \( \varrho_{n,0} = \frac{\gamma_n}{\lambda_{n-1}} = \varrho_n, \ n \geq 1 \). From the assumption ii) and the previous relation, it follows that

\[
\sum_{\nu = n}^{n+2} \theta_{n,\nu} Q_\nu(x) = \Omega_2(x;n)B_n(x) + \zeta_{n,n-1}^{[0]} B_{n-1}(x) + \zeta_{n,n-2}^{[0]} B_{n-2}(x), \quad n \geq 0.
\]

(5.3)

where

\[
\begin{cases}
\lambda_{n+2,n+1} + \theta_{n,n+1} = \beta_{n+1} + \varrho_n, \ n \geq 0, \\
\lambda_{2,0} + \theta_{0,1} + \theta_{0,0} = \gamma_1 + \gamma_2 + \varrho_0, \\
\lambda_{n+2,n} + \theta_{n,n+1} \lambda_{n+1,n} + \theta_{n,n} = \gamma_{n+1} + \gamma_{n+2} (\beta_n + \varrho_n), \ n \geq 1, \\
\theta_{n+1,n} + \lambda_{n,n-1} + \theta_{n,n} \lambda_{n,n-1} = \gamma_n + \gamma_{n-1} + \lambda_{n,n-1}, \ n \geq 2,
\end{cases}
\]

(5.4)

The determinants associated with (5.1) – (5.2) are

\[
\Delta_0(0,2) = \Delta_1(0,2) = 0,
\]

\[
\Delta_n(0,2) = \lambda_{n,n-2} (\lambda_{n+1,n-1} - \gamma_n), \ n \geq 2.
\]

(5.5)

As a consequence of Theorem 3.2, where \( t = 0 \) and \( s = 2 \), we have the following result

\[
A_{n+1}(x) + \varrho_n A_n(x) = B_{n+1}^{[1]}(x) + \varrho_n B_n^{[1]}(x), \quad n \geq 1.
\]

Equivalently,

\[
A_n(x) - B_n^{[1]}(x) = \prod_{\nu = 1}^{n} (\varrho_\nu) (A_1(x) - B_1^{[1]}(x)) = 0, \quad n \geq 1.
\]

But, from (4.15) for \( n = 1 \) we get \( A_1(x) = x - \beta_0 + \frac{\gamma_1}{\varrho_1} \).

From (4.14), with \( n = 0 \), we get \( B_1^{[1]}(x) = x - \beta_0 + \frac{\gamma_1}{\varrho_1} \).

Hence, \( A_n(x) = B_n^{[1]}(x), \ n \geq 0 \). Thus according to (4.15), i) holds.

\[
5. \text{ The case } (t,s) = (0,2)
\]

Let \( \{ B_n \}_{n \geq 0} \) be a MOPS with respect to the linear functional \( u \) and satisfying (1.5). Consider the following finite-type relation between \( \{ B_n \}_{n \geq 0} \) and \( \{ Q_n \}_{n \geq 0} \), with index \( s = 2 \), with respect to \( \phi(x) = 1 \), for \( n \geq 0 \),

\[
Q_n(x) = B_n(x) + \lambda_{n-1} B_{n-1}(x) + \lambda_{n-2} B_{n-2}(x), \quad (5.1)
\]

\[
\exists r \geq 2, \quad \lambda_r, r = 2 \neq 0. \quad (5.2)
\]

From Lemma 2.1, for every system of complex numbers \( \{ \vartheta_{n,\nu} \}_{n=0}^{n+2} \), \( n \geq 0 \), where \( \vartheta_{n,n+2} = 1, \ n \geq 0 \) and \( \vartheta_{r,2} \neq 0 \), there exists a unique MPS \( \{ \Omega_2(x,n) \}_{n \geq 0} \), where \( \Omega_2(x,n) = x^2 + v_{n,1} x + v_{n,0} \), \( n \geq 0 \), and a unique system of complex numbers, \( \{ \lambda_{n,n-1} \}_{n=0}^{n-1} \), \( n \geq 0 \), such that

Proposition 5.1. Let \( \{ B_n \}_{n \geq 0} \) be a MOPS and \( \{ Q_n \}_{n \geq 0} \) be the MPS satisfying (5.1) – (5.2). For every fixed integer \( p \geq 2 \), the following statements are equivalent

i) \( \lambda_n \neq 0, \lambda_{n+1} - \gamma_n \neq 0, \ n \geq p \).
ii) There exist a unique $\text{SCN} (\theta_{n,p}^*)_{n \geq p}$, $n \geq p$, with $\theta_{n,n+2}^* = 1$, $n \geq p$, and $\theta_{r,r}^* \neq 0$, if $p \leq r$, and there exists a unique $\text{MPS} \{\Omega_n^*(x;n)\}_{n \geq p}$, where $\deg \Omega_n^*(x;n) = 2$, $n \geq p$, such that, for $n \geq p$,

$$\Omega_n^*(x;n)B_n(x) = Q_{n+2}(x) + \theta_{n,n+1}^* Q_{n+1}(x) + \theta_{n,n}^* Q_n(x), \quad (5.6)$$

We write

$$\theta_{n,n+1}^* = \frac{\lambda_{n,n+2} - \beta_{n+1} + \lambda_{n+2,n+1} - \lambda_{n-1}}{\lambda_{n,n+2} \lambda_{n+1,n+1} - \lambda_{n-1}} \gamma_n,$$  

$$\theta_{n,n}^* = \frac{\gamma_n}{\lambda_{n,n-2}}.$$

$$\Omega_n^*(x;n) = x^2 + v_{n,1}^* x + v_{n,0}^*, \quad n \geq p, \quad (5.7)$$

where

$$v_{n,0}^* = \theta_{n,n}^* + (\lambda_{n+1,n} - \beta_n) \theta_{n,n+1}^* - \gamma_n - \gamma_n + \lambda_{n+2,n} + \beta_n - \lambda_{n+2,n+1},$$

$$v_{n,1}^* = \theta_{n,n+1}^* - \beta_{n+1} - \beta_n + \lambda_{n+2,n+1}.$$

Example. Let $\{B_n\}_{n \geq 0}$ be the sequence of monic polynomials, orthogonal with respect to the linear functional $u$ such that

$$\langle u, p \rangle = \int_{-\infty}^{+\infty} p(x) e^{-x^2} dx.$$  

This sequence of polynomials was introduced by P. Nevanlinna (see [15]) in the framework of the so-called Freud measures. These polynomials satisfy the three-term recurrence relation (1.5), with coefficients $\beta_n = 0$, $n \geq 0$, and where $\gamma_n$, $n \geq 0$, are given by a non-linear recurrence relation (see [3] and [15])

$$n = 4 \gamma_n (\gamma_n + \gamma_n - \gamma_n), \quad n \geq 1,$$

with $\gamma_0 = 0$ and $\gamma_1 = \Gamma(3/4) \Gamma(1/4)$.

The sequence $\{B_n\}_{n \geq 0}$ satisfies the following structure relation (see [3])

$$B_n^{[1]}(x) = B_n(x) + \lambda_{n,n-2} B_{n-2}(x), \quad n \geq 2, \quad (5.8)$$

where

$$\lambda_{n,n-2} = \frac{4}{n+1} \gamma_n (\gamma_n + \gamma_n - \gamma_n) \neq 0, \quad n \geq 2.$$  

From (5.3), with $Q_n(x) = B_n^{[1]}(x)$, $n \geq 0$, and the fact that the polynomial sequences $\{B_n\}_{n \geq 0}$ and $\{B_n^{[1]}\}_{n \geq 0}$ are symmetric, i.e., $B_n(-x) = (-1)^n B_n(x)$, $n \geq 0$, we get, for $n \geq 0$,

$$B_{n+2}^{[1]}(x) + \theta_{n,n} B_n^{[1]}(x) = \left( x^2 + v_{n,0}^* \right) B_n(x) + \zeta_{n,n-2}^{[0]} B_{n-2}(x), \quad (5.9)$$

where

$$\begin{cases} 
\lambda_{2,0} + \theta_{0,0} = \gamma_1 + v_{n,0}, \\
\lambda_{n+2,n} + \theta_{n,n} = \gamma_{n+1} + \gamma_n + v_{n,0}, \quad n \geq 1, \\
\theta_{n,n} \lambda_{n,n-2} = \gamma_{n-1} + \zeta_{n,n-2}^{[0]}, \quad n \geq 2.
\end{cases} \quad (5.10)$$

Since we have $\lambda_{n,n-2}, \quad n \geq 2$, the choice $\zeta_{n,n-2}^{[0]} = 0$, $n \geq 2$, is possible and yields the inverse relation

$$(x^2 + v_{n,0}^*) B_n(x) = B_{n+2}^{[1]}(x) + \theta_{n,n} B_n^{[1]}(x), \quad n \geq 0, \quad (5.11)$$

where

$$\theta_{n,n} = \frac{n+1}{4 \gamma_{n+1}}.$$

$$v_{n,0}^* = \frac{n+1}{4 \gamma_{n+1}} - \gamma_n - \gamma_{n+1} + \frac{4}{n+3} \gamma_{n+1} \gamma_{n+2} \gamma_{n+3} + 3.$$

Here, the determinants associated with (5.8) are

$$\Delta_n(0,2) = \frac{4}{n+1} \gamma_{n+1} \gamma_{n+2} \left[ \frac{4}{n+2} \gamma_{n+2} \gamma_{n+3} + 1 \right] / \left[ \frac{4}{n+3} \gamma_{n+1} \gamma_{n+2} \gamma_{n+3} + 3 \right]. \quad (5.12)$$

$n \geq 2$, with $\Delta_0(0,2) = \Delta_1(0,2) = 0$.

From Proposition 5.1, we deduce that the uniqueness of the previous inverse relation requires that $\lambda_{n+1,n-1} - \gamma_n = \gamma_n \left[ \frac{4}{n+2} \gamma_{n+2} \gamma_{n+1} - 1 \right] \neq 0$, $n \geq 2$. Equivalently, $\lambda_{n+2,n+1} \neq n+2$, $n \geq 2$. Indeed, by using (5.8), where $n$ is replaced by $n+1$ and taking into account the orthogonality of the polynomial sequence $\{B_n\}_{n \geq 0}$, we get

$$B_{n+1}^{[1]}(x) = x B_n(x) + (\lambda_{n+1,n-1} - \gamma_n) B_{n-1}(x), \quad n \geq 1.$$  

On the other hand, if we suppose that there exists an integer $n_0 \geq 2$ such that $\lambda_{n_0+1,n_0-1} - \gamma_{n_0} = 0$, then $B_{n_0+1}(x) = x B_{n_0}(x)$. In this case (5.11), with $n = n_0$ will be written as $(x^2 + \alpha x + v_{n,0}) B_n(x) = B_{n+2}(x) + \alpha B_{n+1}(x) + \theta_{n,n} B_{n+1}(x)$, for all $\alpha \in \mathbb{C}$. This contradicts the uniqueness of the inverse relation.

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