NONEMBEDDABILITY OF THE KLEIN BOTTLE IN RP³ AND LAWSON'S CONJECTURE

por

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Resumen

Perdomo O.: Nonembedability of the Klein Bottle in RP^3 and Lawson's Conjecture. Rev. Acad. Colomb. Cienc. **29** (110): 149-156, 2005. ISSN 0370-3908.

En 1985 Montiel & Ros demostraron que los únicos toros mínimos en S^3, cuyo primer valor propio del laplacino es 2, son los toros de Clifford. En este artículo demostraremos que es imposible encajar una botella de Klein en el espacio proyectivo 3-dimensional $\mathbf{RP}^3$. Más aún, demostraremos que las únicas superficies cerradas no-orientables que pueden encajarse en $\mathbf{RP}^3$ son aquellas con característica de Euler impar. Después de esto, daremos otra demostración del resultado de Montiel & Ros mencionado arriba, esta vez bajo el supuesto de que el toro en consideración tiene simetría antipodal.

Palabras clave: Botella de Klein, toro de Clifford, espacios proyectivos, superficies mínimas.

Abstract

In 1985 Montiel & Ros showed that the only minimal torus in S^3, for which the first eigenvalue of the Laplacian is 2, is the Clifford torus. Here, we will show first the non-existence of an embedded Klein bottle in $\mathbf{RP}^3$. Indeed we will prove that the only non orientable closed surfaces that can be embedded in $\mathbf{RP}^3$ are those with odd Euler characteristic. Later on, we will give another proof of Montiel & Ros' result, assuming that the minimal torus has $\{x, -x\}$ simmetry. We will also point out that our proof of the non-existence of embedded closed non-orientable surfaces with even Euler characteritic in $\mathbf{RP}^3$, still holds true when we replace $\mathbf{RP}^3$ with a 3-dimensional manifold K constructed in the following way: Let N. be any simply connected 3-dimensional manifold. Let $f : S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3\} \to N$ be an embedding. Let U and V be the two connected components of $N \setminus f(S^2)$. K is the manifold obtained by taking U, and identifying the points in ∂U so that $f(x) = f(-x)$.

Key words: Klein bottle, Clifford torus, projective spaces, minimal surfaces.

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AMS Classification 2000: 53C42 53A10.

Introduction

It is well known that it is impossible to embed a non-orientable closed surface in $\mathbf{R}^3$ (see [5]). However it is possible to embed a projective plane, $\mathbf{RP}^2$, in $\mathbf{RP}^3$. Notice that $\mathbf{RP}^2$ is a non-orientable surface while $\mathbf{RP}^3$ is orientable. In the first part of this paper we prove, in a constructive and simple way, that we cannot embed either the Klein bottle or a Klein bottle with a finite number of handles attached in $\mathbf{RP}^3$.

Minimal hypersurfaces of spheres have been a subject of great importance. They represent critical points of a variational problem, and the study of these hypersurfaces is related to the regularity of the Plateau problem. A first step in this study was to consider surfaces in S^3. The simplest examples of minimal surfaces are the equators, which are surfaces isometric to the set

$$\{(x, y, z, w) \in \mathbf{R}^4 : w = 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1\}$$

and the Clifford torus, which are surfaces isometric to the set

$$\left\{(x, y, z, w) \in \mathbf{R}^4 : z^2 + w^2 = \frac{1}{2} \quad \text{and} \quad x^2 + y^2 = \frac{1}{2}\right\}.$$

In 1966, **Almgren** showed that the only immersed minimal spheres in S^3 are the equators [1]. Even though there are infinitely many ways to minimally immerse a torus in S^3, the only known example that is embedded is the Clifford torus. The conjecture that asserts that the Clifford torus is the only embedded minimal torus in S^3 is known as *Lawson's conjecture*. It is not difficult to prove that, for every immersed closed minimal surface in S^3, 2 is an eigenvalue of the Laplacian operator.

One of the well known conjectures in the study of minimal hypersurfaces of spheres is *Yau's conjecture*. This conjecture, in the case of surfaces, states that if a closed surface in S^3 is embedded and minimal, then 2 is the first eigenvalue of the Laplacian. **Montiel & Ros** showed that for minimal torus in S^3, Yau's conjecture implies Lawson's conjecture [4].

In the second part of this paper, we will use the main theorem of the first part to prove **Montiel & Ros**' result in a shorter way under the additional hypothesis that the minimal torus has antipodal symmetry.

Preliminaries

In this section we will establish some results that we will use to prove our main theorems. Let us start with transversality theory. From linear algebra it is well known that, in general, the intersection of two 2–dimensional subspaces in $\mathbf{R}^3$ is a 1–dimensional space. When we have two surfaces, M_1 and M_2, in a 3-dimensional manifold N, we have, by the implicit function theorem, that if these surfaces satisfy

$$\begin{aligned} &T_m M_1 \cap T_m M_2 \subset T_m N \\ &\text{is } 1\text{−dimensional for every } m \in M_1 \cap M_2 \end{aligned} \tag{1}$$

then the set $M_1 \cap M_2$ is either empty or it is a 1-dimensional submanifold of N. When the condition (1) holds true, we say that M_1 intersects M_2 transversally.

A theorem in tranversality theory gives us,

Theorem 2.1. *Given two smooth surfaces S_1 and M_2 in a 3–dimensional manifold N, it is possible to find a smooth surface $M_1 \subset N$ such that S_1 is homeomorphic to M_1 and $M_1 \cap M_2$ is either empty or a 1–dimensional manifold in N.*

Let us state the following theorem on quotient manifold,

Theorem 2.2. *Let $G \times M \to M$ be a properly discontinuous action of a group G on a differentiable manifold M. The manifold M/G is orientable if and only if there is an orientation of M that is preserved by all the diffeomorphisms of G.*

As a consequence of this theorem we have the following examples: Let us denote by $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$

Example 2.1. Let $G = \{-1, 1\}$ act on S^n by $(-1, x) \to -x$ and $(1, x) \to x$. Clearly this action is properly discontinuous. The diffeomorphism -1 sends the basis $\{v_1, \ldots, v_n\}$ of $T_m S^n$ to the basis $\{-v_1, \ldots, -v_n\}$ of $T_{-m} S^n$. Let us assume that we are taking the orientation on S^n given by the unit normal vector field $\nu(m) = m$. Using this orientation, we have that if $\{v_1, \ldots, v_n\}$ is an oriented basis of $T_m S^n$, then the same basis does not provide an oriented basis of $T_{-m} M$. Therefore, the diffeomorphism -1, which sends the orientation given by a basis $\{v_1, \ldots, v_n\}$ of $T_m S^n$ to the orientation given by the basis $\{-v_1, \ldots, -v_n\}$ of $T_{-m} M$, reverses the orientation on S^n if and only if n is even. Hence $\mathbf{RP}^n = S^n/\{-1, 1\}$ is orientable if and only if n is odd.

Example 2.2. Let M be an embedded torus in S^3 such that if $m \in M$ then $-m \in M$. Let $\nu : M \to S^3$ be a unit normal vector field of M as a submanifold of S^3, i.e. $\nu(m)$ is perpendicular to $T_m M$ and $\nu(m)$ is a vector in T_mS^3. Since M has antipodal symmetry, then $T_mM = T_{-m}M$ for every $m \in M$. Therefore we have that either $\nu(m) = \nu(-m)$ for all $m \in M$ or $\nu(-m) = -\nu(m)$ for all $m \in M$. As we pointed out before, the vector spaces T_mS^3 and $T_{-m}S^3$ are the same but they have different orientations; this implies that if $\nu(m) = \nu(-m)$ then the orientations induced by M on T_mM and $T_{-m}M$ are also different. Since the bases $\{v_1, v_2\}$ and $\{-v_1, -v_2\}$ induce the same orientation on the vector space $\{rv_1 + sv_2 : r, s \in \mathbf{R}\}$, we have that if $\nu(m) = \nu(-m)$, then the manifold $S = M/\{-1, 1\}$ is not orientable. Since the Euler characteristic of M, $\chi(M)$, is twice the Euler characteristic of $S = M/\{-1, 1\}$ and $\chi(M)$ is zero, $S = M/\{-1, 1\}$ is a Klein bottle. The same argument shows that if $\nu(-m) = -\nu(m)$ then $S = M/\{-1, 1\}$ is again a torus.

Nonembeddability of the Klein bottle in $\mathbf{RP}^3$

In this section, we will prove that it is impossible to embed a Klein bottle or a Klein bottle with a finite number of handles attached in the 3-dimensional projective space $\mathbf{RP}^3$. We will achieve this by using some basic criteria to decide when a surface is orientable and by making some constructions in order to estimate the Euler characteristic of any embedded surface in $\mathbf{RP}^3$.

Let us identify $\mathbf{RP}^3$ with the set N of points in $\mathbf{R}^3$ with norm less than or equal to 1 where each point in the *boundary* is identified with its opposite, i.e. if

$$B = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$$
$$\partial B = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$
$$\tau : \partial B \longrightarrow \partial B \ , \quad \tau(m) = -m \ ,$$

then
$$N = B/\{id, \tau\} \ .$$

We may think that we are identifying N with $\mathbf{RP}^3$ using the map
$$\phi : N \to \mathbf{RP}^3$$
given by
$$\phi(x_1, x_2, x_3) = [(x_1, x_2, x_3, \sqrt{1 - x_1^2 - x_2^2 - x_3^2})] \ .$$

Clearly ϕ is well-defined and bijective because antipodal points on the boundary of B are identified.

Notice that under this identification, $\mathbf{RP}^2$ is identified with $\partial B/\{id, \tau\} \subset N$.

Let us denote by $\pi : B \longrightarrow N$ the natural projection, i.e. $\pi(x) = x$ if $|x| < 1$ and $\pi(x) = [x] = \{x, -x\}$ if $|x| = 1$.

Lemma 3.1. *If $M \subset N$ is an embedded surface that intersects transversally $\mathbf{RP}^2$, then the set $C_1 = \pi^{-1}(M \cap \mathbf{RP}^2) \subset S^2 = \partial B$ has one of the following forms* $C_1 = \{\alpha_0, \alpha_1, \overline{\alpha}_1, \ldots, \alpha_k, \overline{\alpha}_k\}$

or $\quad C_1 = \{\alpha_1, \overline{\alpha}_1, \ldots, \alpha_k, \overline{\alpha}_k\}$,

where each α_i and $\overline{\alpha}_i$ is homeomorphic to a circle and $\tau(\alpha_i) = \overline{\alpha_i}$

Proof. Let $M_1 = \pi^{-1}(M) \subset B$. Notice that C_1 is a collection of smooth disjoint closed embedded curves in ∂B because $(M \cap \mathbf{RP}^2)$ is a collection of smooth closed disjoint curves and the map $\pi|_{S^2} : S^2 = \partial B \to \mathbf{RP}^2$ is a covering map. Notice also that M_1 is an embedded surface with boundary in $\mathbf{R}^3$ and $\partial M_1 = C_1$. By the identification made on ∂B we have that if $x \in C_1$ then $-x \in C_1$.

Let us prove by contradiction that there is at most one closed curve contained in C_1 that has antipodal symmetry. Let α_0 and α_0' be two disjoint closed curves contained in C_1, since α_0 is an embedded curve in S^2, it divides S^2 in two simply connected parts U and V; since α_0 has antipodal symmetry, then $\tau(U) = V$, therefore the area of U is the same as the area of V and both area are equal to 2π because the area of the S^2 is 4π; now, since α_0 and α_0' are disjoint, then one of the connected components of $S^2 - \alpha_0'$, let us call it W, is contained in either U or V, this is a contradiction because the area of W is 2π. Since there can only be one closed curve with antipodal symmetry in C_1 we have that there are two possibilities for the set C_1

Case 1: If C_1 contains a circle α_0 which is invariant under τ, then

$$C_1 = \{\alpha_0, \alpha_1, \overline{\alpha}_1, \ldots, \alpha_k, \overline{\alpha}_k\} \ ,$$

where the α_i's are closed curves, $\tau(\alpha_0) = \alpha_0$ and $\tau(\alpha_i) = \overline{\alpha}_i$, for $i = 1, 2, ..., k$.

Case 2: If C_1 does not contain a circle which is invariant under τ, then

$$C_1 = \{\alpha_1, \overline{\alpha}_1, \ldots, \alpha_k, \overline{\alpha}_k\} \ ,$$

where the α_i's are closed curves, and $\tau(\alpha_i) = \overline{\alpha}_k$ for $i = 1, 2, ..., k$. ∎

Theorem 3.1. *If M is a closed surface in $\mathbf{RP}^3$ that intersects transversally $\mathbf{RP}^2$ and $C_1 = \pi^{-1}(M \cap RP^2) \subset S^2 = \partial B$ contains a closed curve which is invariant under the antipodal map, then the Euler characteristic of M is odd.*

Proof. In the same way we did before, let us identify $\mathbf{RP}^3$ with $N = B/\{id, \tau\}$ and $\mathbf{RP}^2$ with $\partial B/\{id, \tau\}$.

Since there is a closed curve in C_1 invariant under the antipodal map, we have, by Lemma 3.1, that

$$C_1 = \{\alpha_0, \alpha_1, \overline{\alpha}_1, \ldots, \alpha_k, \overline{\alpha}_k\} = \{\beta_0, \beta_1, \ldots, \beta_{2k}\}$$

with $\tau(\alpha_i) = \bar{\alpha}_i$.

For $i = 1, ..., 2k$, let B_i be the connected component with smaller area of $S^2 \setminus \beta_i$. Let M_2 be the manifold that is obtained by gluing $2k+1$ disks to M_1, one along each β_i. It is not difficult to see that we can embed M_2 in $\mathbf{R}^3$, e.g. we can make this gluing in $\mathbf{R}^3$ by choosing disks of the form

$$\{rx : 1 \leq r \leq r_i \quad \text{and} \quad x \in \beta_i\} \cup \{r_i x : x \in B_i\} ,$$

for $i = 1, \ldots, 2k$, and the last disk of the form

$$\{rx : 1 \leq r \leq r_0 \quad \text{and} \quad x \in \beta_0\} \cup \{r_0 x : x \in V\} ,$$

where V is one of the connected components of $S^2 \setminus \alpha_0$.

Let us denote by $\chi(S)$ the Euler characteristic of a surface S. Let us take a triangulation of M such that each circle $\pi(\alpha_i) \subset M$ contains exactly 3 edges of the triangulation. Let F be the number of faces, E the number of edges and V the number of vertices of the triangulation. Clearly this triangulation induces a triangulation on $M_1 = \pi^{-1}(M)$, the number of faces, edges and vertices for this triangulation on M_1 is F, $E+3k+3$ and $V+3k+3$, respectively. This happens because the circle α_0 contains now 6 edges and 6 vertices instead of the 3 edges and 3 vertices of $\pi(\alpha_0)$ and, for $i = 1, \ldots, k$, the 3 edges and 3 vertices of $\pi(\alpha_i)$ give us 3 edges and 3 vertices in α_i and $\overline{\alpha}_i$. Therefore, $\chi(M) = \chi(M_1)$. Now taking this triangulation on M_1, we define a new triangulation on M_2 by adding $6+2k$ new triangles to the triangulation defined on M_1 in this way:

(i) The disk attached to the circle α_0 is thought as 6 triangles with 6 vertices in the boundary and one vertex in the interior of the disk. The gluing is taken so that vertices on the boundary of the glued disk are identified with the 6 vertices of α_0.

(ii) Each disk attached to an α_i (or $\overline{\alpha}_i$) is considered as a single triangle and the gluing is taken so that the 3 vertices of the boundary of the glued disk are identified with the 3 vertices of α_i (or $\overline{\alpha}_i$).

Having made these considerations, it is not difficult to see that the new triangulation is going to have $6+2k$ more faces, 6 more edges and 1 more vertex than the triangulation on M_1, Therefore,

$$\chi(M_2) = \chi(M_1) + 2k + 1 = \chi(M) + 2k + 1 .$$

Since M_2 is closed and can be embedded in $\mathbf{R}^3$, we have that M_2 is orientable. Thus, its Euler characteristic is even. This observation, together with the equation above, implies that the Euler characteristic of M must be odd. ∎

Theorem 3.2. *If M is a closed surface in $\mathbf{RP}^3$ that intersects transversally $\mathbf{RP}^2$ and $C_1 = \pi^{-1}(M \cap \mathbf{RP}^2)$ does not contain a circle which is invariant under the antipodal map, then M is orientable.*

Proof. Since none of the closed curves in C_1 is invariant under τ, then

$$C_1 = \{\alpha_1, \bar{\alpha_1}, \ldots, \alpha_k, \bar{\alpha_k}\}$$

with $\tau(\alpha_i) = \overline{\alpha}_i$. Let us assume that $k = 1$. Let K_1^ϵ be the set of points in $M_1 = \pi^{-1}(M)$ that are within a distance ϵ of α_1 and let K_2^ϵ be the set of points in M_1 that are within a distance ϵ of $\overline{\alpha}_1$, we will assume that ϵ has been chosen so that K_i^ϵ are smooth surfaces homeomorphic to cylinders. Let M_2 be the closed surface obtained by gluing to M_1 a cylinder Σ embedded in $\mathbf{R}^3 \setminus B$, the boundary of this glued cylinder is $\alpha_1 \cup \overline{\alpha}_1$. Let $\phi : [0,1] \to \alpha_1$ be a regular parametrization of α_1. Notice that the map ϕ is homotopic to the map $\tau \circ \phi$ in Σ (this is the key observation in this proof). Therefore we can define a parametrization

$$\psi : [0,1] \times [0,1] \to \Sigma$$

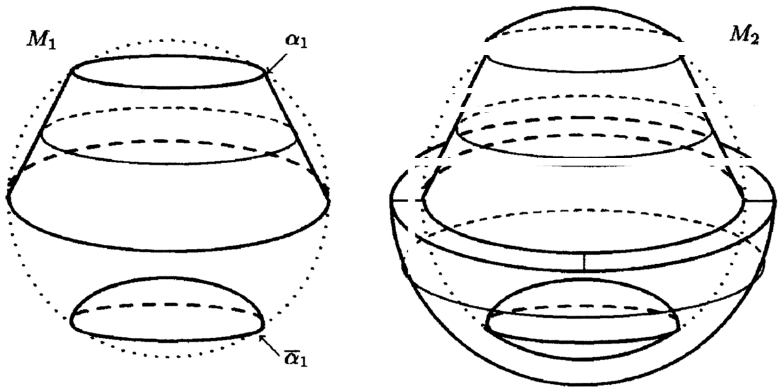

Figure 1

In this figure $M \subset \mathbf{RP}^3$ is homeomorphic to $\mathbf{RP}^2$ and it intersects $\partial B/\{-1,1\}$ in two circles. The inverse image of these circles under $\pi : B \to \mathbf{RP}^3$ is the union of three circles. One of them, the equator, is invariant under the antipodal map. $M_1 = \pi^{-1}(M)$ is the union of a cylinder and a disk; M_2 is the union of two spheres. In this case $r_1 = r_2 = 1$, $r_0 > 1$ and V is the south hemisphere.

such that

$$\psi(0,v)=\psi(1,v)\ ,\psi(u,0)=\phi(u) \text{ and } \psi(u,1)=\tau\circ\phi(u).$$

We will prove that M is homeomorphic to M_2. This would imply that M is orientable because M_2 is orientable. Recall that a closed surface can be embedded in $\mathbf{R}^3$ only if it is orientable. Let $\Sigma_1=\psi([0,1]\times[0,1/2])$ and $\Sigma_2=\psi([0,1]\times[1/2,1])$. Let γ_i $i=1,2$ be two closed curves in M_1 such that $\partial K_1^\epsilon=\alpha_1\cup\gamma_1$ and $\partial K_2^\epsilon=\overline{\alpha}_1\cup\gamma_2$. Since Σ_i is homeomorphic to K_i^ϵ, then, we can define two homeomorphisms φ_1 and φ_2 such that

$$\begin{aligned}\varphi_1:\Sigma_1\to K_1^\epsilon\ ,\quad &\varphi_1(\psi(u,1/2))=\psi(u,0)=\phi(u)\ ,\\ &\varphi_1(\psi(u,0))\in\gamma_1\ ,\\ \varphi_2:\Sigma_2\to K_2^\epsilon\ ,\quad &\varphi_2(\psi(u,1/2))=\psi(u,1)=\tau\circ\phi(u)\ ,\\ &\varphi_2(\psi(u,1))\in\gamma_2\ .\end{aligned}$$

Since the manifolds M_1 and $M_1\setminus K_1^\epsilon\cup K_2^\epsilon$ are homeomorphic we can define a homeomorphism φ_3 such that

$$\varphi_3: M_1\to M_1\setminus K_1^\epsilon\cup K_2^\epsilon$$

such that

$$\varphi_3(\psi(u,0))=\varphi_1(\psi(u,0)) \text{ and } \varphi_3(\psi(u,1))=\varphi_2(\psi(u,1)).$$

Using the maps φ_1, φ_2 and φ_3, we can define our homeomorphism ξ from M_2 to M in the following way:

$$\begin{aligned}\xi(m)=&[\varphi_1(m)] \quad \text{if } m\in\Sigma_1\ ,\\ \xi(m)=&[\varphi_2(m)] \quad \text{if } m\in\Sigma_2\ ,\\ \xi(m)=&[\varphi_3(m)] \quad \text{if } m\in M_1\ .\end{aligned}$$

The map ξ is a continuous well defined map because of the conditions imposed on the maps φ_i, $i=1,2,3$ on the boundary. Therefore M is homeomorphic to M_2 which is orientable. When $k>1$, let B_i be the connected component with smaller area of $S^2\setminus\alpha_i$ for $i=1,\dots,k$. In the case the sets $B_1,\dots B_k$ are disjoint, let us take k disjoint curves, $\omega_1,\dots,\omega_k$, contained in the closure of the set $\mathbf{R}^3\setminus B$ such that each ω_i connects a point $p_i\in B_i$ with the point $-p_i$. Let M_2 be the surface obtained by gluing to $M_1=\pi^{-1}(M)$ k cylinders $\Sigma_1,\dots,\Sigma_k$. These cylinders are chosen so that the boundary of each Σ_i is the union of α_i with $\overline{\alpha}_i$ and the bounded component of the closed surface $\Sigma_i\cup B_i\cup\tau(B_i)$ contains the curve ω_i, in other words, Σ_i is the cylindrical part of the boundary of the solid obtained after thickening the curve ω_i. The same procedure that we did in the case $k=1$ shows that the surface M is homeomorphic to the surface M_2, since M_2 is embedded in $\mathbf{R}^3$, then M_2 is orientable, hence M is also orientable. The proof in the general case is essentially the same, we glue $\Sigma_1,\dots,\Sigma_k$ cylinders to the surface M_1 to obtain an orientable surface that is homeomorphic to M. In this general case, the cylinders can be chosen so that every time $B_i\subset B_j$ then, Σ_i is contained in the bounded component of the closed surface $\Sigma_j\cup B_j\cup\tau(B_i)$.

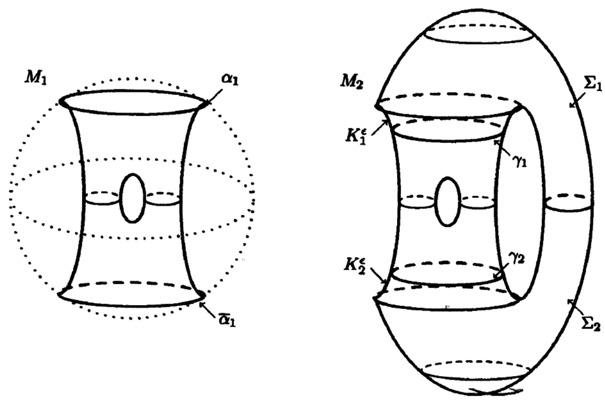

Figure 2

In this figure $M\subset\mathbf{RP}^3$ is homeomorphic to a double torus and it intersects $\partial B/\{-1,1\}$ in one circle. The inverse image of this circle under $\pi:B\to\mathbf{RP}^3$ is the union of two circles. $M_1=\pi^{-1}(M)$ is a cylinder with a handdle attached. M_2 is again homeomorphic to a double torus.

Theorem 3.3. *A closed non-orientable surface M can be embedded in $\mathbf{RP}^3$ if and only if the Euler characteristic of M is odd.*

Proof. The set $M=\{[(x,y,0)]:x^2+y^2\le 1\}\subset N=B/\{id,\tau\}=\mathbf{RP}^3$ shows an embedding of $\mathbf{RP}^2$ in $\mathbf{RP}^3$, clearly we can attach as many handles as we want in a embedded fashion to M in a neighborhood of the point $(0,0,0)$. This shows that every non-orientable closed surface with odd Euler characteristic can be embedded in $\mathbf{RP}^3$.

Now, let S be a non-orientable surface embedded in $\mathbf{RP}^3$. By Theorem 2.1, there exists an embedded surface $M\subset\mathbf{RP}^3$ homeomorphic to S, that intersects transversally $\mathbf{RP}^2=\partial B/\{-1,1\}$, by Lemma 3.1, Theorem 3.1 and Theorem 3.2, we have that the Euler characteristic of M must be odd, therefore the Euler characteristic of S must be also odd. ∎

Corollary 3.1. *It is impossible to embed a Klein bottle, or any closed non-orientable closed surface with even Euler characteristic, in $\mathbf{RP}^3$.*

Using the same method that we used to prove Theorem 3.3, we can prove the slightly more general result:

Theorem 3.4. *Let N be any simply connected 3 dimensional manifold. Let*

$$f:S^2=\{(x_1,x_2,x_3)\in\mathbf{R}^3:x_1^2+x_2^2+x_3^2=1\}\to N$$

be an embedding. Let U and V be the two connect components of $N\setminus f(S^2)$. If K is the manifold obtained by taking U, and identifying the points in ∂U so *that $f(x) = f(-x)$, then a closed non-orientable surface M can be embedded in K if and only if the Euler characteristic of M is odd.*

Proof. Let K_1 be the manifold obtained by gluing the unit ball B in $\mathbf{R}^3$ to U using the map f. All the arguments made to prove Theorem 3.1 to Theorem 3.3 work by replacing $\mathbf{R}^3$ by the simply connected manifold K_1, and $\mathbf{R}^3 \setminus B$ by the set B view as a subset of K_1. Notice also that every non orientable surface embedded in K must intersect the surface $\mathbf{RP}^2 = f(S^2)/\{1, \tau\}$ where τ, in this case, is the map from $f(S^2)$ to $f(S^2)$ defined by $\tau(p) = f(-f^{-1}(p))$. ∎

A partial result on Lawson's conjecture

Let us start this section with some well known facts about minimal surfaces in S^3. If $\psi : M \to S^3$ is an immersed minimal surface in S^3 and $\psi(m) = (x_1(m), x_2(m), x_3(m), x_4(m))$ then the minimallity of M is equivalent to the condition $\Delta x_i = -2x_i$ on the functions $x_i : M \to S^3$, $i = 1, \dots, 4$.

Let us denote by $\nu : M \to S^3$, and

$$\nu(m) = (\nu_1(m), \nu_2(m), \nu_3(m), \nu_4(m)) ,$$

be the unit normal vector field on M as a submanifold of S^3. The shape operator at $m \in M$ is the symmetric linear operator $A_m : T_mM \to T_mM$ defined by $A_m(v) = -d\nu_m(v)$.

By the Codazzi equations and the minimallity of M, we have that the functions $\nu_i : M \to S^3$ satisfy the equation $\Delta\nu_i = -|A|^2\nu_i$ for $i = 1, \dots, 4$. Here $|A|^2(m) = |A_m(e_1)|^2 + |A_m(e_2)|^2$, where $\{e_1, e_2\}$ is any orthonormal basis of T_mM. Notice that,

$$\begin{aligned}|A|^2(m) &= |A_m(e_1)|^2 + |A_m(e_2)|^2 = |d\nu_m(e_1)|^2 + |d\nu_m(e_2)|^2\\ &= |(d(\nu_1)_m(e_1), d(\nu_2)_m(e_1), d(\nu_3)_m(e_1), d(\nu_4)_m(e_1))|^2 +\\ &\qquad |(d(\nu_1)_m(e_2), d(\nu_2)_m(e_2), d(\nu_3)_m(e_2), d(\nu_4)_m(e_2))|^2\\ &= \sum_{i=1}^{4}(d(\nu_i)_m(e_1)^2 + d(\nu_i)_m(e_2)^2) = \sum_{i=1}^{4}|\nabla\nu_i|^2(m) .\end{aligned}$$

The eigenvalues $\kappa_1(m)$, $\kappa_2(m)$ of A_m are known as the principal curvatures of M at $m \in M$. The Gauss equation applied to surfaces in $\mathbf{R}^3$ gives us that the Gauss curvature is the product of the principal curvatures.

For surfaces in S^3 the Gauss equation gives us that the Gauss curvature is the product of the principal curvatures plus 1, i.e. if $K(m)$ is the Gauss curvature of $M \subset S^3$ at $m \in M$, we have

$$K(m) = 1 + \kappa_1(m)\kappa_2(m) \qquad (2) .$$

Since M is minimal, $\kappa_1(m) + \kappa_2(m) = 0$. Thus

$$|A|^2(m) = \kappa_1^2(m) + \kappa_2^2(m) = 2\kappa_1^2(m) ,$$

and therefore the equation (2) can be written as

$$K(m) = 1 - \frac{|A|^2}{2} .$$

In the case that M is topologically a torus, the Gauss Bonnet formula gives us that $\int_M K = 0$ or equivalently

$$\int_M |A|^2 = \int_M 2 = 2 \text{ times the area of } M \qquad (3) .$$

Now we are ready to prove the main theorem in this section.

Theorem 4.1. *If M is an embedded minimal surface in S^3 which is invariant under the antipodal map and such that the first eigenvalue of the Laplacian is 2, then M is the Clifford torus.*

Proof. Let $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ be the unit normal vector. By Corollary 3.1 and Example 2.2 we have that $\nu(-m) = -\nu(m)$ for all $m \in M$ otherwise $M/\{-1, 1\}$ will define an embedded Klein bottle in $\mathbf{RP}^3$. Therefore the function $\nu_i : M \to \mathbf{R}$, $i = 1, \dots, 4$ are odd functions and $\int_M \nu_i = 0$ i.e. they are functions perpendicular to the constant function $f \equiv 1$ viewed as elements of the Hilbert space $L^2(M)$. Now, since we are assuming that the first eigenvalue of the Laplacian is 2, we have

$$\int |\nabla\nu_i|^2 \geq 2\int_M \nu_i , \quad i = 1, 2, 3, 4 , \qquad (4)$$

with equality if and only if $\Delta\nu_i = -|A|^2\nu_i = -2\nu_i$. (5)

Notice that if the equation (5) holds true for $i = 1, \dots, 4$, then $|A|^2(m) = 2$ for all $m \in M$. Summing up the inequalities in (4) above from $i = 1$ to $i = 4$ we get

$$\int |A|^2 \geq \int_M 2 \qquad (6)$$

but the equation (3) gives us that we have an equality in the equation (6) instead of an inequality, therefore, we have an equality in each of the equations in (4), hence, $|A|(m) = 2$ for all $m \in M$. This last equation implies that M must be a Clifford torus by the main result in [2]. ∎

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Recibido el 5 de julio de 2004

Aceptado para su publicación el 9 de febrero de 2005