OVERVIEW ON MODELS IN HOMOTOPICAL ALGEBRA

by

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To the memory of Professor Jairo Charris Castañeda

Resumen

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Un funtor covariante $\Delta \to \mathcal{A}$ se dice un *objecto modelo de $\mathcal{A}$*. Los objetos modelo producen en $\mathcal{A}$ un tema de estudio muy parecido a la topología algebraica cuando $\mathcal{A}$ es la categoría de los espacios topológicos. En este trabajo se describen los escenarios en los cuales se desarrollan estos conceptos y las principales resultados desarrollados por el autor sobre objetos modelos.

Palabras clave: Categoría modelo, objetos simpliciales, cosimpliciales, homotopía, levantamiento.

Abstract

A covariant functor $\Delta \to \mathcal{A}$ is called a *model object of $\mathcal{A}$*. Model objects produce in $\mathcal{A}$ a subject matter very much as algebraic topology when $\mathcal{A}$ is the category of topological spaces. Here we describe the settings on which such concepts are developed and describe the main features developed by the author about model objects.

Key words: Model category, Simplicial objects, Cosimplicial, Homotopy, Lifting.

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1. Model, closed model, and pre-model categories

By a model category (Daniel Quillen [QD67]) we mean a category $\mathcal{A}$ together with three distinguished classes of morphisms F (fibrations), C (cofibrations) and WE (weak equivalences) such that:

M.0. *Axiom of admissibility:* $\mathcal{A}$ is closed under finite projective and inductive limits.

M.1. *Lifting axiom:* Given a solid arrow diagram in $\mathcal{A}$, where i is a cofibrations and p is a fibration and where either i or p is a weak equivalence, then the dotted arrow exists.

M.2. *Factorization axiom:* Any morphism f in $\mathcal{A}$ can be factored $f = pi$ where i is a cofibration and weak equivalence and p is a fibration. Also $f = pi$ where i is a cofibration and p is a fibration and a weak equivalence.

M.3. F is closed under composition, base change and any isomorphism is a fibration. C is closed under composition, cobase change and any isomorphism is a cofibration.

M.4. The base extension of a morphism which is a fibration and weak equivalence is a weak equivalence. The cobase extension of a map which a cofibration and weak equivalence is a weak equivalence.

M.5. *Triangular axiom:* If in a commutative diagram

two of the morphisms are weak equivalences so is the third.

If $(\mathcal{A}, F, C, WE)$ is a model category, then there exists an associated category, called the *homotopy category of* $(\mathcal{A}, F, C, WE)$ denoted simply by $H_\circ \mathcal{A}$ which is the localization of $\mathcal{A}$ with respect to WE. $H_\circ \mathcal{A}$ is characterized by the existence of a functor $r : \mathcal{A} \to H_\circ \mathcal{A}$ and the following universal property of $(r, H_\circ \mathcal{A})$: for every $f \in WE$, $r(f)$ is an isomorphism. If there exists another pair $(F, \mathcal{B})$, $F : \mathcal{A} \to \mathcal{B}$ such that for every $f \in WE$, $F(f)$ is an isomorphism then, there exists a unique functor $\rho : H_\circ \mathcal{A} \to \mathcal{B}$ such that $\rho r = F$.

Note that in general, if $r(f)$ is an isomorphism, f need not be a weak equivalence. There is however, a special kind of model category in which $r(f)$ is an isomorphism if, and only if f is a weak equivalence. They are called *closed model categories.* Before we define them, we give some notation: In a model category a morphism which is both a fibration and a weak equivalence is called a trivial fibration. TF denotes the class of such morphisms. A morphism which is both a cofibration and a weak equivalence is called a trivial cofibration, TC denotes the class of such maps. We call the classes F, C, TC, TF and WE the *classes of structural maps* of the model category.

From our point of view, the main feature of closed model categories is that, with the exception of WE, the classes of structural maps are characterized by lifting properties: PM2 in next definition.

On the other hand one can also have "almost" model categories which fail to be model because WE fails to behave well. They are "pre model categories".

By a *pre-model category*, (Roberto Ruiz, [RR76]) we mean a category $\mathcal{A}$ together with five classes of maps F, C, TF, TC and WE such that:

P.M.1. $TF \subseteq F$ and $WE = TF \circ TC$ (i.e. a map is a weak equivalence if and only if it factors as a trivial cofibration followed by a trivial fibration).

P.M.2. F, C, TF, TC admit the following characterization by liftings.
   i f is a fibration if and only if f has right lifting property with respect to TC.
   ii f is a trivial fibration if and only if f has right lifting property with respect to C.
   iii If f has left lifting property with respect to TF, then f is a cofibration.
   iv If f has left lifting property with respect to F, then f is trivial cofibration.

P.M.3 Any map f admits two factorizations:

where h is a cofibrations and i is a trivial fibration and h' is a trivial cofibration and i' is a fibration.

Remarks:

i i and ii implies that iii and iv in $P.M.2$ become equivalences.

ii Any isomorphism belong to each one of the classes F, C, TF, TC (P.M.2) and hence to WE.

iii F, C, TF, TC are closed under composition and retracts.

iv $TC \subseteq C$, and furthermore $TF = F \cap WE$ and $TC = C \cap WE$.

v Let

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\bar{f} \downarrow & & \downarrow f \\
Z & \longrightarrow & K
\end{array}
$$

be a Cartesian square in $\mathcal{A}$. If f is a fibration (resp. trivial fibration) so is $\bar{f}$. Let

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
f \downarrow & & \downarrow \underline{f} \\
Z & \longrightarrow & K
\end{array}
$$

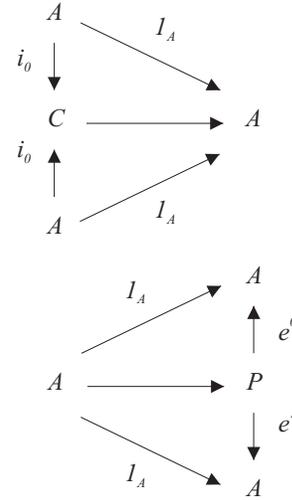be a cocartesian square in $\mathcal{A}$. If f is a cofibration (resp. a trivial cofibration) so is $\underline{f}$.

vi $F \cap C \cap WE = $ isomorphisms of $\overline{\mathcal{A}}$.

Thus a pre model category have good behavior in liftings (as good as in closed model categories) but bad behaved WE: it only goes up to containing all isomorphisms and factorization $WE = TF \circ TC$.

The "closure" of a model category is define as follows [RR77]: given a model category $(\mathcal{A}, F, C, WE)$ there exists a unique pre model category $(\mathcal{A}, \overline{F}, \overline{C}, \overline{TF}, \overline{TC}, \overline{WE})$ such that $F \subseteq \overline{F}$, $C \subseteq \overline{C}$, $TF \subseteq \overline{TF}$, $TC \subseteq \overline{TC}$ and $WE \subseteq \overline{WE}$. This pre model category measures the extent in which a model category is a closed model category. It is proved that when the original model category is closed, it coincides with its closure. The closure can also be viewed as "the theory of liftings" of the model category. The uniqueness of the closure is implied by the following fact, meaningful in itself. If $\mathfrak{A}$ is a class of morphisms in $\mathcal{A}$ and we denote by $[\mathfrak{A}]$ the class of all retracts of members of $\mathfrak{A}$, then one has that $\overline{F} = [F]$, $\overline{C} = [C]$, $\overline{TF} = [TF]$, $\overline{TC} = [TC]$.

Hence a model category is closed if and only if the classes F, C, TF, TC (or equivalently F, C, WE) are closed under retracts. In pre model, model and closed model categories a "cylinder" object (resp. "path" object) of A is a diagram as the next first (resp. second)

one.

$$
\begin{array}{ccc}
A & & \\
i_0 \downarrow & \searrow^{1_A} & \\
C & \longrightarrow & A \\
i_0 \uparrow & \nearrow^{1_A} & \\
A & &
\end{array}
$$

$$
\begin{array}{ccc}
& \nearrow^{1_A} & A \\
A & \longrightarrow P & \uparrow e^0 \\
& \searrow_{1_A} & \downarrow e^1 \\
& & A
\end{array}
$$

C is the "actual" cylinder object of A, i_j $(j=0,1)$ are trivial cofibrations (which are usually inclusions) and P is the actual "path object" of A and C^j are trivial fibrations (which usually are evaluation functions). Then one has for $f, g : A \to B$ that f is "left homotopic" to g if there exists a cylinder of A and $h : C \to B$ such that $h \circ i_0 = f$ and $h \circ i_1 = g$. The dual procedure provides "right homotopy". If the map $\phi \to A$ (ϕ initial object) belongs to C (A is a "co fibrant object") then left homotopy is an equivalence relation on $Hom(A, B)$ and left homotopy implies right homotopy [QD67]. Since dual assertions hold, right and left homotopy coincide and are equivalence relations on $Hom(A, B)$ when A is co fibrant and B is fibrant.

Elsewhere homotopy is done through homotopy systems: Let $\mathcal{A}$ be a category. A homotopy system (Kan, [KD55,58]) $Z = (I, J_0, J_1, q)$ consists of the following:

i A "cylinder" covariant functor $I : \mathcal{A} \to \mathcal{A}$.

ii Three natural transformations $J_0 : 1_{\mathcal{A}} \to I$, $J_1 : 1_{\mathcal{A}} \to I$, $q = I \to 1_{\mathcal{A}}$ such that $qJ_0 = qJ_1 = 1$.

Homotopy is then given as follows: Let $f, g : X \to Y$ be morphisms and $\mathcal{A}$. We say that f is homotopic to g, denoted $f \cong g$, if there exists a morphism $\rho : I(X) \to Y$ in $\mathcal{A}$ such that $\rho \circ J_0(X) = f$ and $\rho \circ J_1(X) = g$.

The homotopy relation as defined is not in general an equivalence relation. But it is reflexive and compatible with composition: if $f, g : X \to Y$, $h : Y \to Z$, $k : K \to X$ then $f \simeq g$, implies that $hf \simeq hg$ and $fk \simeq gk$.

If $\sim$ denotes the equivalence relation generated by $\simeq$, then one has the following definition: Let $f : X \to Y$ a morphism in $\mathcal{A}$. We say that f is a homotopy equivalence if $[f] \in \mathcal{A}(X, Y)/\sim$ is an isomorphism in Mor $\mathcal{A}/\sim$.

Fibrations and cofibrations for a homotopy system are given in Kamps [KK69]. We will provide now examples of homotopy systems. The most common ones are given in categories with final object different from the initial one, if it exists. We will denote by $*$ the final object and by $\emptyset$ the initial one.

If $\mathcal{A}$ is a category, then say that $\mathcal{A}$ is *pointed* if $* \cong \emptyset$. Otherwise is *unpointed*. Let $\mathcal{A}$ be unpointed. For an object A of $\mathcal{A}$ the morphisms $* \to A$ are called the *points* of A and $\mathcal{A}(*, A)$ is called the *underlying set* of A. In unpointed categories there are plenty of homotopy systems. In fact, Let $\mathcal{A}$ be an unpointed category closed for finite products. Let X be an object of $\mathcal{A}$ and x_0, x_1 (if that order) points of X. Then there exists a natural isomorphism $\iota : 1_A \to 1_A \times *$, and the bimap $I : \mathcal{A} \to \mathcal{A}$, $A \mapsto A \times X$, $f \mapsto f \times 1_X$ is a covariant functor. Further, $d_i(A) : A \overset{\iota_A}{\to} A \times * \overset{1_A \times X_i}{\to} A \times X$, for $i = 0, 1$ and $\pi_1 : A \times X \to A$ are natural transformations on $A \in Obj\mathcal{A}$ and $\pi_1 \circ d_i(A) = 1_A$. Thus (I, d_0, d_1, s) is a homotopy system on $\mathcal{A}$.

Normal homotopies in the categories of topological spaces Top and simplicial sets $\Delta^\circ S$ are of this kind, the first one induced by $\{I = [0, 1], 0, 1\}$ and the second by $\{\Delta[1], \epsilon^0, \epsilon^1\}$ where $\epsilon^i : [0] \to [1]$, $0 \mapsto i$ induce simplicial points (still denoted by ϵ^0, ϵ^1) on the second. We assume the reader familiar with them.

Let $Z = (I, J_0, J_1, q)$ be a homotopy system in $\mathcal{A}$. We say that a map $F : E \to B$ is a *fibration* (or a Z fibration) if f has the right lifting property with respect to J_0 i.e. to the class of maps $J_0(X) : X \to I(X)$, $X \in Obj\mathcal{A}$. A map $i : A \to X$ is called a (Z) cofibration if for any commutative diagram.

$$
\begin{array}{ccc}
 & \overset{J_0(A)}{\longrightarrow} & \\
A & & I(A) \\
\downarrow i & & \downarrow \\
X & \underset{f}{\longrightarrow} & Y
\end{array}
$$

there exists a homotopy $\overline{\varphi} : I(X) \to Y$ such that $\overline{\varphi}I(i) = g$ and $\overline{\varphi}J_0(X) = f$.

One has the following properties for fibrations and cofibrations in a homotopy system [KK69]: Isomorphisms are fibrations and cofibrations. Projections are fibrations. Fibrations are closed under composition, base extension, and retracts. Cofibrations are closed under composition, co-base extension, and retracts.

2. Simplicial Systems, category, functor

We change a little Quillen's version of simplicial categories [QD67] to the notion of *simplicial systems* in a given category. The reason is that, as we will see, there may be more than one way in which a category is a simplicial category. For us then a "simplicial category" will be a pair formed by a category and a "simplicial system".

Let $\mathcal{A}$ be a category. By a *simplicial system* in $\mathcal{A}$ we mean a functor $\underline{Hom}_\mathcal{A} : \mathcal{A} \times \mathcal{A} \to \Delta^\circ S$ such that the following conditions hold:

S.1. For any objects X, Y, Z of $\mathcal{A}$ there exists a "composition" (simplicial) map:

$$\underline{Hom}_\mathcal{A}(X, Y) \times \underline{Hom}_\mathcal{A}(Y, Z) \to \underline{Hom}_\mathcal{A}(X, Z)$$

level wise denoted by $(f, g) \mapsto g \circ f$, which is associative in the sense that, if $f \in \underline{Hom}_A(X, Y)_n$, $g \in \underline{Hom}_A(Y, Z)_n$, and $h \in \underline{Hom}_A(Z, K)_n$, then $(h \circ f) \circ g = h \circ (f \circ g)$.

S.2. There exists a natural isomorphism

$$\lambda : \mathcal{A}(\ , \ ) \to [\underline{Hom}_\mathcal{A}(\ , \ )]_0$$

denoted by $\mathcal{A}(X, Y) \to (\underline{Hom}_\mathcal{A}(X, Y))_0$; $u \mapsto \widetilde{u}$ such that if $u \in A(X, Y)$, $f \in \underline{Hom}_\mathcal{A}(Y, Z)_n$ and $g \in \underline{Hom}_\mathcal{A}(W, X)_n$ then

$$s_0^n(u) = \underline{Hom}_\mathcal{A}(u, Z)_n(f)$$

and

$$s_0^n(u) \circ g = \underline{Hom}_\mathcal{A}(W, u)_n(g)$$

where (abusing notation) s_0^n denotes the composite of the (in general different) functions

$$\underline{Hom}_\mathcal{A}(X, Y)_p \overset{s_0}{\to} \underline{Hom}_\mathcal{A}(X, Y)_{p+1}$$

for $p = 0, 1, \ldots, n - 1$

By a *simplicial category* we mean a pair $(\mathcal{A}, \underline{Hom}_\mathcal{A})$ where $\mathcal{A}$ is a category and $\underline{Hom}_\mathcal{A}$ is a simplicial system on $\mathcal{A}$. As for functors among them:

Let $(\mathcal{A}, \underline{Hom}_\mathcal{A})$ and $(B, \underline{Hom}_\mathcal{B})$ be two simplicial categories. By a *simplicial functor*

$$F : (A, \underline{Hom}_\mathcal{A}) \to (B, \underline{Hom}_\mathcal{B})$$

we mean a functor $F : \mathcal{A} \to \mathcal{B}$ together with maps

$$\underline{Hom}_\mathcal{A}(X,Y) \to \underline{Hom}_\mathcal{B}(F(X), F(Y)); \ f \mapsto F(f)$$

such that $F(f \circ g) = F(f) \circ F(g)$ and $F(\widetilde{u}) = \widetilde{F(u)}$. We say that F is *strictly simplicial* if the maps

$$\underline{Hom}_\mathcal{A}(X,Y) \to \underline{Hom}_\mathcal{B}(F(X), F(Y))$$

define a natural transformation (denoted again by F).

$$F : \underline{Hom}_\mathcal{A} \to \underline{Hom}_\mathcal{B}(F \times F)$$

Since the simplicial functors that we will use are always strictly simplicial we will talk simply of "simplicial functors" and refer to the "strict" part only when specially necessary.

Simplicial categories have cylinders and path objects: Let $(\mathcal{A}, \underline{Hom}_\mathcal{A})$ be a simplicial category. Let X be an object of $\mathcal{A}$ and K a simplicial set.

i By a cylinder object associated to (X, K) we mean a pair $(X \otimes K, \alpha)$ where $X \otimes K$ is an object of $\mathcal{A}$ and

$$\alpha : K \to \underline{Hom}_\mathcal{A}(X, X \otimes K)$$

is a simplicial map such that for each Y in $\mathcal{A}$ the simplicial map

$$\varphi : \underline{Hom}_\mathcal{A}(X \times K, Y) \to (\underline{Hom}_\mathcal{A}(X,Y))^K$$

next defined is an isomorphism: φ_n has domain $\underline{Hom}_\mathcal{A}(X \otimes K, Y)_n$, codomain $\Delta^\circ S(K \times \Delta[n], \underline{Hom}_\mathcal{A}(X,Y))$ and $\rho \mapsto (\boldsymbol{o}) \circ (\alpha \times 1) \circ (1 \times \underline{\rho})$. More explictly the image of ρ is the composition of the maps: $1 \times \underline{\rho}$ with domain $K \times \Delta[n]$, and codomain $K \times \underline{Hom}_\mathcal{A}(X \times K, Y)$; $\alpha \times 1$ has domain $K \times \underline{Hom}_\mathcal{A}(X \times K, Y)$, and codomain $\underline{Hom}_\mathcal{A}(X, X \otimes K) \times \underline{Hom}_\mathcal{A}(X \otimes K, Y)$; $\boldsymbol{o}$ with domain $\underline{Hom}_\mathcal{A}(X, X \otimes K) \times \underline{Hom}_\mathcal{A}(X \otimes K, Y)$, and codomain $\underline{Hom}_\mathcal{A}(X,Y)$ where

$$\underline{\rho} : \Delta[n] \to \underline{Hom}_\mathcal{A}(X \otimes K, Y)$$

is the simplicial map associated to ρ, namely the unique simplicial map such that $\underline{\rho}(l_{[n]}) = \rho$.

ii By a path object associated to (X, K) we mean a pair (X^K, β) where X^K is an object of $\mathcal{A}$ and $\beta : K \to \underline{Hom}_\mathcal{A}(X^K, X)$ is a simplicial function such that the induced map.

$$\psi : \underline{Hom}_\mathcal{A}(Y, X^K) \to \underline{Hom}_\mathcal{A}(Y, X)^K$$

described below is an isomorphism: ψ_n with domain $\underline{Hom}_\mathcal{A}(Y, X^F)_n$, and codomain

$$(K \times \Delta[n], \underline{Hom}_\mathcal{A}(Y, X)),$$

where

$$\rho \mapsto (\mathrm{o}) \circ (Pr_2, Pr_1) \circ (1 \times \underline{\rho})$$

i.e. the image of ρ is the composition of the maps: $1 \times \rho$ with domain $K \times \Delta[n]$, and codomain $K \times \underline{Hom}_\mathcal{A}(Y, X^K)$; (Pr_1, Pr_2) with domain $K \times \underline{Hom}_\mathcal{A}(Y, X^K)$, and codomain $\underline{Hom}_\mathcal{A}(Y, X^K) \times \underline{Hom}_\mathcal{A}(X^K, X)$; $\boldsymbol{o}$ with domain $\underline{Hom}_\mathcal{A}(Y, X^K) \times \underline{Hom}_\mathcal{A}(X^K, X)$, and domain $\underline{Hom}_\mathcal{A}(Y, X)$

Let $(\mathcal{A}, \underline{Hom}_\mathcal{A})$ be simplicial category. Let X be an object of $\mathcal{A}$ and K, L simplicial sets. There are canonical isomorphisms $X \otimes (K \times L) \overset{\simeq}{=} (X \otimes K) \otimes L$ and $(X^K)^L \overset{\simeq}{=} X^{K \otimes L}$ when all of the objects involved are defined.

Recall that the homotopy relation on simplicial sets in general is not an equivalence relation. We denote by $\simeq$ the homotopy relation in $\Delta^\circ S$ and by $\sim$ the equivalence relation induced by $\simeq$. Recall further that the functional simplicial set associated to spaces X and Y (where the idea of simplicial categories was taken from) is the simplicial set $Hom(X, Y)$ whose n-th level is given as follows: $\underline{\Delta}$ denotes the cosimplicial space with $\underline{\Delta}^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1\}$ the co faces $\delta^i : \underline{\Delta}^n \to \underline{\Delta}^{n+1}$ is the function that adds 0 in the i-th coordinate and $\rho^i : \underline{\Delta}^n \to \underline{\Delta}^{n-1}$, $x_i + x_{i+1}$ in the i-th coordinate. $\underline{\Delta}$ will be called "the standard model" in Top. Now, back to $Hom(X, Y)$ we take

$$\underline{Hom}(X, Y)_n = Top(\underline{\Delta}^n \times X, Y)$$

faces and degeneracies induced by those of $\underline{\Delta}$. It is clear that

$$\underline{Hom}(X, Y)_0 \overset{\simeq}{=} Top(X, Y)$$

and is well known that two maps $f, g : X \to Y$ are homotopic in Top if as members of $\underline{Hom}(X, Y)_0$ they are homotopic. The generalization of this situation to simplicial categories is as follows: Let $(\mathcal{A}, \underline{Hom}_\mathcal{A})$ be a simplicial category, $f, g : X \to Y$ be morphisms in $\mathcal{A}$. We say that f is strictly homotopic to g, and denote it $f \simeq g$, if their images $\overline{f}, \overline{g} \in \underline{Hom}_\mathcal{A}(X, Y)_0$ are homotopic i.e. if $\overline{f} \simeq \overline{g}$. We say that f is homotopic to g, denoted $f \sim g$ if $\overline{f} \sim \overline{g}$ in $\underline{Hom}_\mathcal{A}(X, Y)$. We denote $\pi_0(\underline{Hom}_\mathcal{A}(X, Y))$ by $\pi_0(X, Y)$. The category $\pi_0 \mathcal{A}$ is defined as having as objects those of $\mathcal{A}$ and for each pair X, Y of $Obj\mathcal{A} = (Obj\pi_0\mathcal{A})$.

$$\pi_0\mathcal{A}(X, Y) = \pi_0[\underline{Hom}_\mathcal{A}(X, Y)]$$

with composition induced by the one in $\mathcal{A}$.

The existence of path and cylinder objects provides a nice representation of $\underline{Hom}_\mathcal{A}(X, Y)$. In fact if

$(\mathcal{A}, \underline{Hom}_\mathcal{A})$ is a simplicial category and $\mathcal{B}$ is a subcategory of $\Delta^\circ S$, then we say that $\mathcal{A}$ *admits a cylinder through* $\mathcal{B}$ if there exists a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{A}$ such that for each $X \in \mathcal{A}$ and $K \in \mathcal{B}$ the image of (X, K) is a cylinder object of X in $\mathcal{A}$, say

$$\begin{array}{ccc} & X & \\ d^1 \downarrow & & \searrow^{1_X} \\ X \otimes K & \xrightarrow{s} & X \\ d^0 \uparrow & & \nearrow \\ & X & \nearrow_{1_X} \end{array}$$

We say that $\mathcal{A}$ admits paths through $\mathcal{B}$ if there exists a functor $\mathcal{A} \times \mathcal{B}^\circ \to \mathcal{A}$ such that for each $X \in \mathcal{A}$ and $K \in \mathcal{B}$ the image of (X, F) is a path object of X in $\mathcal{A}$ say

$$\begin{array}{ccc} & & X \\ & \nearrow^1 & \uparrow D_0 \\ X & \xrightarrow{s} & X^K \\ & \searrow & \downarrow D_1 \\ & & X \end{array}$$

Now we consider $\mathbf{\Delta} : \Delta \to \Delta^\circ S$, the standard models of $\Delta^\circ S$. On it $\mathbf{\Delta}_n = \Delta[n]$ is the simplicial set with $\Delta[n]_m$ the set of increasing functions $[n] \to [m]$ where $[p] = \{0, 1, ..., p\}$. $d_i : \Delta[n]_m \to \Delta[n]_{m-1}$ maps $\alpha : [m] \to [n]$ in $\alpha \circ \delta_i$ where $\delta_i : [m-1] \to [m]$ is the $1-1$ and increasing function which misses i in $[m]$. Further $\rho_j : \Delta[n]_m \to \Delta[n]_{m+1}$ maps α in $\alpha \circ \rho_j$ where $\rho_j : [m+1] \to [m]$ is the onto increasing function which repeats j in $[m]$. $\Delta[n]$ is thus a simplicial set. The $\Delta[n]$'s form a cosimplicial object of $\Delta^\circ S$ whose n-th level is $\Delta[n]$, $d_p^i : \Delta[n]_p \to \Delta[n+1]_p$ sends α in $\delta_i \circ \alpha$ and $s_p^j : \Delta[n]_p \to \Delta[n-1]_p$ sends α in $\rho_j \circ \alpha$. One thus have that $\Delta[n]$, $d^i : \Delta[n] \to \Delta[n+1]$ and $s^j : \Delta[n] \to \Delta[n-1]$ conform a cosimplicial object of $\Delta^\circ S$, "the model of the simplicial $\Delta[n]$'s".

We denoted by $\mathbf{\Delta}$ again the image $\mathbf{\Delta}(\Delta)$ which is a subcategory of $\Delta^\circ S$.

Suppose that $\mathcal{A}$ admits cylinders through $\mathbf{\Delta}$. Then for each $X \in \mathcal{A}$ there exist a composition functor $\Delta \to \mathcal{A} \times \mathbf{\Delta} \to \mathcal{A}$ with $[n] \mapsto X \otimes \Delta[n]$ and

$$(w : [n] \to [m]) \mapsto (1_X \otimes w^X : X \otimes \Delta[n] \to X \otimes \Delta[m])$$

This composition is a cosimplicial object of $\mathcal{A}$. Similarly if $\mathcal{A}$ admits paths through $\mathbf{\Delta}$, then for each $X \in \mathcal{A}$ one

has a simplicial object of $\mathcal{A}$, $\Delta \to A \times \mathbf{\Delta} \to \mathcal{A}$ with $[n] \mapsto (X, \Delta[n]) \mapsto X^{\Delta[n]}$ and

$$(w : [n] \to [m]) \mapsto (1_X, w^*) \mapsto (X^{w^*} : X^{\Delta[m]} \to X^{\Delta[n]}).$$

One uses these two object to prove the following.

Suppose $\mathcal{A}$ admits cylinders through $\mathbf{\Delta}$. Then for each pair X, Y of objects of $\mathcal{A}$, $\underline{Hom}_\mathcal{A}(X,Y)$ is (up to isomorphism) the simplicial set whose n-th level is $\underline{Hom}_\mathcal{A}(X,Y)_n = \mathcal{A}(X \otimes \Delta[n], Y)$. Suppose $\mathcal{A}$ admits paths through $\mathbf{\Delta}$. Then for each X, Y objects of $\mathcal{A}$, $\underline{Hom}_\mathcal{A}(X,Y)$ is (up to isomorphism) the simplicial set whose n-th level is

$$\underline{Hom}_\mathcal{A}(X,Y)_n = \mathcal{A}(X, Y^{\Delta[n]})$$

Therefore in case $\mathcal{A}$ admits cylinders through $\mathbf{\Delta}$, homotopy in $\mathcal{A}$ is a left homotopy. For $f, g : X \to Y$ morphisms of $\mathcal{A}$, $f \sim g$ if and only if there exists a morphism $H : X \otimes \Delta[1] \to Y$ such that $H \circ d^1 = f$ and $H \circ d^0 = g$.

Similarly when $\mathcal{A}$ admits paths through $\mathbf{\Delta}$, homotopy in $\mathcal{A}$ is a (right) homotopy: $f \sim g$ if and only if there exists a morphism $T : X \to Y^{\Delta[1]}$ in $\mathcal{A}$ such that $d_1 \circ T = f$ and $d_0 \circ Tg$.

Of course when $\mathcal{A}$ admits path and cylinders through $\mathbf{\Delta}$, homotopy in $\mathcal{A}$ is given by the (then) equivalent ways above.

We present now in some detail the most typical example of a "model object" at work which at long last produces a model category. Then we introduce its generalization to "model object" (or simply "model") and show that they produce pre model categories. In order to deal with the difference (from model to pre model categories) we work the missing parts (homotopy) by means of the homotopy system of its simplicial system.

3 Kan Fibrations and Trivial Fibrations

Let X be a simplicial set. Let $n \in N^*$. By a (simple) $n-box$ we mean a subset $\{x_0, \ldots, \hat{x}_k, \ldots, x_n\}$ of X_n, (where $\hat{x}_k$ means that there is no element indexed on k), such that for each pair $i, j = 0, 1, 2, \ldots, n+1$, if $i < j$ then $d_i x_j = d_{j-1} x_i$. By a *trivial $n-box$* of X we mean a subset $\{x_0, \ldots, x_{n+1}\}$ of X such that for each pair $i, j = 0, 1, \ldots, n+1$ with $i < j$ then $d_i x_i = d_{j+1} x_i$.

The equalities $d_i x_j = d_{j-1} x_i$ for simple and trivial boxes will be refereed to as the *compatibility relations* of the x_i's. They are fulfilled emptily for $n = 0$. So

$0-$boxes of X are subsets of X_0 with one element, and trivial $0-$boxes, pair (x_0, x_1), x_0 not necessarily different from x_1.

Let $S = \{x_0, \ldots, \hat{x}_k, \ldots, x_n\}$ (respectively, $S' = \{x_0, \ldots, x_{n+1}\}$). We say that x *is a filler* of S (resp. of S') if for each $i \neq k$, $d_i(x) = x_i$ (resp. for each i, $d_i(x) = x_i$).

If $f : X \to Y$ is a simplicial function and $S = \{x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}\}$ is a box in X (resp. $S' = \{x_0, \ldots, x_{n+1}\}$ is a trivial $n-$box in X) then $f(S) = \{f(x_0), \ldots f(x_{n+1})\}$ is an $n-$box in Y, (resp. $f(S')$ is a trivial $n-$box in Y).

Let $f : X \to Y$ be a simplicial map. f is said to be a *Kan fibration* if given any $n-$box S such that the image box $f(S)$ admits a filler $y \in Y_{n+1}$, then there exists a filler $x \in x_{n+1}$ of S, such that $f_{n+1}(x) = y$. f is said to be a trivial fibration if f_0 is onto and if given a trivial $n-$box S in X such that $f(S)$ admits a filler $y \in Y_{n+1}$, then there exists a filler $x \in x_{n+1}$ of S such that $f(x) = y$.

A simplicial set X is called a *Kan complex* (also is said to be (Kan) *fibrant*, or to have *Kan extension condition*) if the simplicial function $X \to *$ (where $*$ is any simplicial point) is a Kan fibration.

Let X be a simplicial set and $x \in X_n$. Recall that there exists a unique simplicial map $\widetilde{x} \colon \Delta[n] \to X$ such that, $\widetilde{x}\ (1_{[n]}) = x$. It is clear that if $f : X \to Y$ is a simplicial map, then $f \circ \widetilde{x} = \widetilde{f(x)}$. One also have that $\widetilde{x} \circ d^i = \widetilde{d^i(x)}$ and $\widetilde{x} \circ s^j = \widetilde{s_j(x)}$.

We call a set $\{a_0, \ldots \hat{a}_k, \ldots, a_j\} = S$ with $a_j : \Delta[n] \to X$ a *functional $n-$box* in X if $a_j \circ d^i = d^{j-1} \circ a_i$, for $i < j$ $i, j \neq k$. When a_k is not omitted we call the set a *trivial functional $n-$box* in X. If $X \overset{f}{\to} Y$, then we denote by $f(S) = \{f \circ a_0, \ldots, f \overset{\wedge}{\circ} a_k, \ldots, f \circ a_{n+1}\}$ the functional $n-$box *image* of S.

We call $\Delta[n+1] \overset{a}{\to} X$ a filler of S if $a \circ d^i = a_j$ for $i \neq k$. Similarly for a trivial functional $n-$box.

Fibrations and trivial fibrations can be given by means of functional boxes: f is a Kan fibration if and only if for each functional $n-$box S whose image $f(S)$ admits a filler b, there exists a filler a of S such that $f \circ a = b$. f is a trivial fibration if and only if f_0 is onto and for each functional trivial $n-$box S whose image $f(S)$ admits a filler b there exists a filer a of S

such that $f \circ a = b$. One can produce trivial boxes from standard ones. A simple but tedious proof can be supplied for the following: if $S = \{x_0, \ldots, \hat{x}_k, \ldots x_{n+1}\}$ is an $n-$box ($n \geq 1$) in a simplicial set X, then the set $\{\underline{x}_0, \ldots, \underline{x}_n\} \subseteq X_{n-1}$ defined by

$$\underline{x}_i = \begin{cases} d_{k-1}(x_i) & \text{if } i < k \\ d_k(x_{i+1}) & \text{if } i \geq k \end{cases}$$

is a trivial $n-1$ box in X, which we denote by $d_k(S)$.

If $S = \{x_0, \ldots, \hat{x}_k, \ldots x_{n+1}\}$ is a trivial $n-$box, such that $d_k(S)$ admits a filler x, then

$$\{x_0, \ldots, x_{k+1}, \ldots x_{n+1}\}$$

is a trivial $n-$box.

It can be seen that any trivial fibration is a Kan fibration. Then one have a pre model category as follows: F is the class of Kan fibrations, TF is the class of trivial fibrations, C (resp. TC) is the class of maps with left lifting property for TF (resp. F) and WE (weak equivalences) is $TF \circ TC$. Quillen [QD67] proves that in fact F, C, WE is a closed model structure in $\Delta^{\circ}S$, that is $(\Delta^{\circ}S, F, C, WE)$ is a closed model category.

The use of maps of the kind $\Delta[n] \to X$ on the theorems allows us to generalize the concepts of fibrations, trivial and Kan complexes changing maps $\Delta[n] \to X$ to maps $Y^n \to X$ where $Y : \Delta \to \Delta^{\circ}S$ is a covariant functor. The version by boxes of the kind $\{x_0, \ldots, \hat{x}_k, \ldots x_{n+1}\}$ and $\{x_0, \ldots, x_{n+1}\}$ will be useful in order to relate the generalization through Y with the standard theory.

Now we provide the generalization of the concept of fibrations to get fibrations and trivial fibrations associated to a functor $Y : \Delta \to \mathcal{A}$.

We reestablish in detail the general definitions so as to point out how the standard theory is in fact a particular case of the theory induced by Y. Before we complete the pre model category associated to Y we give the counterpart (for Y) of the theorems in the previous paragraph of fibrations and trivial fibrations (when $\underline{\mathbf{\Delta}}$ was used).

Next we show how these two concepts, Y fibrations and Y trivial fibrations, can be characterized (as well as some of their properties) by the use of the singular functor $S_Y : \mathcal{A} \to \Delta^{\circ}S$ associated to Y. Finally we complete the pre model category associated to Y.

4. *Y* fibrations and *Y* Trivial Fibrations

The generalizations of the concepts of boxes, trivial boxes and fillers can be given as follows.

Definition:

i Let X be an object of $\mathcal{A}$. By a *simple Y^n box (or a Y^n box)* in X we mean a family of maps $\{\, a_0, \dots, \hat{a}_k, \dots, a_{n+1}\,\}$, $0 \le k \le n + 1$, where $a_i : Y^n \to X$ is such that if $i < j$, $i, j \ne k$, then $a_j \circ d^i = a_i \circ d^{j-1}$, $d^i, d^{j-1} : Y^{n-1} \to Y^n$.

ii By a *trivial Y^n box* for $n \ge 1$ we mean a family $\{a_0, \dots, a_{n+1}\}$ of maps $a_i : Y^n \to X$ such that $a_j \circ d^i = a_i \circ d^{j-1}$ if $i < j$, then $a_j \circ d^i = a_i \circ d^{j-1}$. A trivial Y^0 box is a family $\{a_0, a_1\}$, $a_i : Y^0 \to X$.

iii Let $S = \{a_0, ..., \hat{a}_k, ..., a_{n+1}\}$ be a Y^n box (resp. $S' = \{a_0', \dots a_{n+1}'\}$ be a trivial Y^n box). By a *filer* of S (resp. S') we mean a map $a : Y^{n+1} \to X$ such that for each $i \ne k$ (resp. for each i) the following diagram commutes

If $f : X \to Y$ is a simplicial map and if $S = \{a_0, \dots, \hat{a}_k, \dots, a_{n+1}\}$ is a Y^n box in X then it is clear that $\{fa_0, \dots, f\hat{a}_k, \dots, fa_{n+1}\}$ is a Y^n box in Y. Similarly the Y^n box (resp. trivial Y^n box) of the fa_i's will be called the image of S by f and will be denoted in both the simple and trivial case by $f(S)$.

We require Y to have the following property : Let K_{n+1} be any set formed with at least $n-1$ of the maps $d^i : Y^n \to Y^{n+1}$ in $\mathcal{A}$. Then for any K_{n+1} there exists an object B on $\mathcal{A}$ and a map $j : B \to Y^{n+1}$ such that each equation $j \circ X = d^i$ has solution (denoted $d_i|$) and further if B' and $j' : B' \to Y^{n+1}$ admit solutions to $j' \circ X = d^i$ then there exists a unique $H : B \to B'$ such that $j' \circ H = j$.

Thus j is unique up to isomorphism. When in K_{n+1} the map d^k is missing we denote j by $i : Y[n+1, k] \to Y^{n+1}$ and when $K_{n+1} = \{d^0, d^1, \dots, d^n\}$, j is denoted by $i : \delta Y^{n+1} \to Y^{n+1}$.

When $d^i : Y^n \to Y^{n+1}$ have underlying functions (and Y^n underlying set) then

$$Y[n+1, k] = \bigcup_{i \ne k} d^i(Y^n) \text{ and } \delta Y^{n+1} = \bigcup_{i=0}^{n} d^i(Y^n)$$

Of course in general there exist a unique

$$i_k : Y[m+1, k] \to \delta Y^{n+1}$$

such that (abusing the notation) $i \circ i_k = i$.

i Let $f : X \to K$ be a map in $\mathcal{A}$. We say that f *is a Y fibration* if given any Y^n box S such that $f(S)$ admits a filler b, then there exists a filler a of S such that $fa = b$.

ii f is said to be a *Y trivial fibration* if for any trivial Y^n box whose image $f(S)$ admits a filler b, there exists a filler a of S such that $fa = b$. Furthermore f has "the Y^0 lifting property" i.e. for any $b : Y^0 \to K$ there exists $a : Y^0 \to X$ such that $fa = b$.

iii X is *Y Kan complex* (resp. *F trivial complex*) if $X \to *$ is a Y fibration (resp. Y trivial fibration).

Consider a family $\{a_0, \dots, \hat{a}_k, \dots, a_{n+1}\} = S$, $a_i : Y^n \to X$ ($n \ge 1$). Then S is a Y^n box (resp. a trivial Y^n box) iff there exists a map (filler)

$$Y[n+1, k] \xrightarrow{a} X \text{ (resp. } a : \delta Y^{n+1} \to X)$$

such that for each $i \ne k$ then $a \circ d^i| = a_i$. The family $\{a_0, ..., a_{n+1}\}$ is a trivial Y^n box if there exits a map $a : \delta Y^n \to X$ such that $a \circ d^i = a_i$ for each i.

The effect on Y fibrations and Y trivial fibrations is the following:

Let $f : X \to K$ be a map on $\mathcal{A}$. Then f is a Y fibrations (resp. Y trivial fibration) iff f has right lifting property with respect to inclusions of the kind $Y[n, k] \xrightarrow{i} Y^n$, $n \ge 1$ and $0 \le k \le n$. (resp. $\delta Y^n \to Y^n$, $n \ge 0$).

Note that X is a Y Kan complex iff every Y^n box admits a filer. Furthermore X is an F trivial complex iff every trivial Y^n box admits a filler.

Recall that given $Y : \Delta \to \mathcal{A}$, there exists associated to it the singular functor $S_Y : \mathcal{A} \to \Delta^\circ S$, given by $(S_Y(X))_n = \mathcal{A}(Y^n, X)$ which gets faces and degeneracies from the ones of Y by composition, and if $f : X \to K$ then $S_Y(f)$ is the simplicial functions whose n-th level is given by $(S_Y(f))_n(a) = fa$. Note that $S_Y(X)$ is a simplicial set and that the elements of $(S_Y(X))_n$ are the kind of maps $Y^n \to X$ we have used to define Y^n boxes and trivial Y^n boxes.

Similarly fillers on Y^n boxes, trivial of otherwise, are fillers in the corresponding simplicial set $S_Y(X)$. It is rather simple to verify also, that $f : X \to K$ has the Y^0

lifting property if and only if $(S_Y(f))_0$ is onto. One gets then the following results.

Let $f : X \to K$ be a $\mathcal{A}$ map: f is a Y fibration if and only if $S_Y(f)$ is a Kan fibration. f is a Y trivial fibration if and only if $S_Y(f)$ is a trivial fibration. Finally, any Y trivial fibration is a Y fibration.

5. Completing the pre model category associated to Y

Now we complete the Y structure on $\mathcal{A}$. To Kan fibration and trivial fibration one adds the following definition:

i A map $f : X \to K$ is called a Y trivial cofibration if it has left lifting property with respect to the class of Y fibrations.

ii f is called a Y cofibration if it has left lifting property with respect to the class of Y trivial fibration.

iii f is called a Y weak equivalence if it factors as $j \circ h$ where h is a Y trivial cofibration and j is a Y trivial fibration.

From these classes of maps we are primarily concerned with fibrations, cofibrations and weak equivalences.

In what follows we will give conditions under which this five classes of maps form a pre model structure in $\mathcal{A}$. Thus far we have:

i The following classes of maps are closed under composition and contain all of the isomorphism of $\mathcal{A}$: Y-fibrations, Y trivial fibrations, Y cofibrations, Y trivial cofibrations.

ii The base extension of a map which is a Y fibration (resp. Y trivial fibration) is again a Y fibration (resp. a Y trivial fibration) The co base extension of a map which is a Y cofibration (resp. a Y trivial cofibration) is again a Y cofibration (resp. a Y trivial cofibration).

iii The classes of Y fibration, Y trivial fibration, Y cofibration, and Y trivial cofibrations are closed under retracts.

iv Suppose a map $X \xrightarrow{f} Y$ factors as $k \circ h$ with h a Y trivial cofibration and k a Y fibration (resp. with h a Y cofibration and k a Y trivial fibration). If f has right lifting property for Y trivial cofibrations, then f is a Y fibration. (resp. if f has right lifting property for Y cofibrations, then f is a Y trivial fibrations).

If the factorization axiom holds for the Y structure then: f is a Y fibration if and only if it has right lifting property for Y trivial cofibrations. Also f is a Y trivial fibration if and only if it has right lifting property with respect to Y cofibration.

As we have mentioned in a model category the prefix "trivial" has special meaning which for the Y structure is still valid. In fact f is a Y trivial fibration if and only if f is a Y fibration and a Y weak equivalence. Also f is a Y trivial cofibration if and only if is a Y cofibration and Y weak equivalence.

Thus so far the following axioms for model and closed model hold for the classes of structural maps associated to Y; for model categories: $M.0$, $M.1$, $M.3$, $M.4$. For closed model categories: $C.M.1$, $C.M.4$ and partially $C.M.3$: $Y - F$, $Y - TF$, $Y - C$ are closed under retracts.

Now we want to give conditions on Y so that the factorization axiom hold. From the remark of proposition 2.24 it will follow as well that the classes of structural maps associated to Y form a pre model category.

6. Smallness and the factorization axiom

The concept of smallness that we use here is actually the sequential one used by Quillen [QD67] as well as the procedure to build up factorization of maps whenever there exists a family $\{A_i \to B_i\}$ with the A_i small. A more refined version can be found in RR94.

Let A be an object of $\mathcal{A}$. We say that A is *small* if

$$Hom(A, \varinjlim Z^m) = \varinjlim Hom(A, Z^m)$$

for any family $\{Z^m\}$, $m \in \mathbb{N}$. More precisely, A is small if for any sequence $Z^0 \to Z^1 \to Z^2 \to \ldots$ of maps and for any $\alpha : A \to \lim Z^n$, there exits $m \in \mathbb{N}$ and a map $\underline{\alpha} : A \to Z^n$ such that $j \circ \underline{\alpha} = \alpha$.

Let $Y : \Delta \to \mathcal{A}$ be a covariant functor. We say that Y is small if for each $n \in \mathbb{N}^*$, Y^n is small.

The cosimplicial character of Y implies smallness of some subobjects of the Y^n's whenever Y is small. In fact let $I \subseteq [n] = \{0, 1, \ldots, n\}$. Then $Y[n, I]$ is small. In particular $Y[n, k]$ and δY^n are small.

Notice also that if A is small and B is retract of A in the usual sense i.e. there are maps $B \xrightarrow{\rho} A$ and $A \xrightarrow{\eta} B$ such that $\eta\rho = 1_B$, then B is also small.

Here are some examples of small Y's.

i $\mathbf{\Delta}$ is obviously small i.e. for each $n \in M$, $\Delta[n]$ is small. Hence so are $\Delta[n,k]$ and $\delta\Delta[n]$.

ii If Y is small then for any $n \in N$, the object obtained by dropping the first n levels and last n cofaces and co-degenacies, $RC^n(Y)$, is small. Hence in particular $RC^{(n)}(\mathbf{\Delta})$ is small for any $n \in N$.

iii If Y is small then the functor $R_Y : \Delta^\circ S \to \Delta^\circ S$, left adjoint of $S_Y : \Delta^\circ S \to \Delta^\circ S$, send small objects into small object. That follows since S_Y commutes with sequential direct limits. Even more: Y is small if and if S_Y commutes with sequential direct limits.

We establish now the general situation implied by smallness about factorization of arrows.

Proposition: Let $A = \{A_i \to B_i \mid i \in I\}$ be a family of maps in $\mathcal{A}$. If for each $i \in I$, A_i is small, then any map $f : X \to Y$ factors as $f = k \circ h$ where h and k have the following properties.

i h has the left lifting property for the class of maps with right lifting property for A.

ii k has right lifting property for A.

Thus if $Y : \Delta \to \mathcal{A}$ is small then any map $f : X \to K$ factors in two ways: $f = k \circ h$ with $h \in Y - C$ and $k \in Y - TF$ and also as $f = k' \circ h'$ with $h' \in Y - TC$ and $k' \in Y - F$. That is to say, if Y is small, then factorizations axiom holds for the $Y-$structure. Also f is a Y fibration if and only if f has right lifting property for $Y - TC$ and f is a Y trivial fibration if and only if f has right lifting property for $Y - C$.

Hence we also have the following: when Y is small, $\mathcal{A}$ together with the classes (or structural maps) $Y - F$, $Y - TF$, $Y - C$, $Y - TC$ and $Y - WE$ form a pre model category.

7. The Homotopy System Associated to Y

Although in Kan [KD55,561] the development of homotopy groups associated to homotopy system is done through cubical complexes, we have found easier to use simplicial sets and their standard homotopy groups in order to associate homotopy groups to a homotopy. In fact, as we will see, Y homotopy as we define it will become, for a suitable Y, a "simplicial homotopy", namely the homotopy associated to the Y simplicial system (a concept studied in next paragraph) in which the machinery of standard homotopy of simplicial sets is available. Here, however, we develop Y homotopy independently of the Y simplicial system.

In this and the next paragraph we assume that Y is a "pointed" model of $\mathcal{A}$, that is to say Y^0 is the final object of $\mathcal{A}$.

Note that the following defines a homotopy system in $\mathcal{A}$:

i $I : \mathcal{A} \to \mathcal{A}$; $X \mapsto X \times Y^1$; $\alpha \mapsto \alpha \times 1_{Y^1}$.

ii $J_i : 1_{\mathcal{A}} \to I$; $X \mapsto J_i(X) : X \stackrel{\cong}{=} X \times Y^0 \stackrel{1 \times d^i}{\to} X \times Y^1$.

iii $g : I \to 1_{\mathcal{A}}$; $X \mapsto g(X) : X \times Y^1 \stackrel{1 \times S^o}{\to} X \times Y^0 \stackrel{\cong}{=} X$.

We call it the Y *homotopy system in* $\mathcal{A}$ and the corresponding homotopy is the Y homotopy. If $f, g : X \to K \in \mathcal{A}$, and f is homotopic to g through this homotopy we write $f \stackrel{Y}{\sim} g$, $f \sim g(Y)$ or when the use of Y^1 is to be emphasized we write $f \stackrel{Y^1}{\sim} g$.

Therefore $f \stackrel{Y}{\sim} g$ if and only if here exits $H : X \times Y^1 \to K$ such that $H \circ d^0 = f$ and $H \circ d^1 = g$.

As for the effect of $S_Y : \mathcal{A} \to \Delta^\circ S$ itself on Y homotopy, if $f \stackrel{Y}{\sim} g$, then $S_Y(f) \sim S_Y(g)$, where $\sim$ denotes the standard homotopy in $\Delta^\circ S$. However we will present a structure which permits the use of simplicial homotopy in full power in favor and the one in $\mathcal{A}$ induced and Y, including homotopy groups, exact homotopy sequences, etc, as done by Quillen [QD67].

8. The Y Simplicial Structure of $\mathcal{A}$

The Y simplicial structure is, roughly speaking, the generalization of functional complexes in $\Delta^\circ S$, when in $(X^K)_n = \Delta^\circ S(K \times \Delta[n], X)$ the models $\Delta[n]$ are substituted by Y^n, where $Y : \Delta \to \Delta^\circ S$.

The formal definition of simplicial system in categories, and the concept of simplicial category was given in the introduction. Here study first the machinery available in the simplicial system induced by special functors $Y : \Delta \to \mathcal{A}$, then the relations with the Y structure and Y homotopy. In this paragraph we consider a pointed Y which realizes products on standard models. Therefore its realization $R_Y : \Delta^\circ S \to \mathcal{A}$ commutes with finite products. On the other hand the following assignments define a functor $Hom_Y : \mathcal{A} \times \mathcal{A} \to \Delta^\circ S$:

i For $X, K \in \mathcal{A}$, $Hom_Y(X, K) = S_{Y \times \overline{X}}(K)$.

ii For $f : X^1 \to X$ and $g : K \to K^1$, $Hom_Y(f, g)$ is the map given level wise by

$$Hom_Y(f, g)_n : Hom_Y(X, K)_n \to Hom_Y(X^1, K^1)_n$$

which maps $\alpha \mapsto g \circ \alpha \circ (1_{Y^n} \times f)$.

Notice that $Hom_Y(X,K)$ is a generalization of K^X. In fact $Hom_Y(X,K)_n = S_{Y \times \overline{X}}(K)_n = \Delta^\circ S(Y^n \times X, K)$.

We show next that Hom_Y admits an associated simplicial composition, as required in a simplicial systems.

i The following defines a simplicial functor for any simplicial sets X, K, Z:
$Hom_Y(X,K)_n \times Hom_Y(X,Z)_n \to Hom_Y(X, ZK)_n$
$(\alpha, \beta) \mapsto \beta \circ \alpha = \beta \circ (\alpha \times 1_{Y^n}) \circ (1 \times \Delta(Y^n))$
Or more graphically $\beta \circ \alpha$ is the following composition

$$X \times Y^n \overset{1 \times \Delta[n]}{\to} X \times Y^n \overset{\alpha \times 1_{Y^n}}{\to} K \times Y^n \overset{\beta}{\to} Z$$

ii The composition of part i is associative in the sense that for X, K, Z, T and $f \in Hom_Y(X,K)_n$, $g \in Hom_Y(Z,K)_n$ and $h \in Hom_Y(Z,T)_n$ then $(h \circ f) \circ g = h \circ (f \circ g)$.

To complete the simplicial system we are required to show that $Hom_Y(X,K)$ is a simplicial set built up on $\Delta^\circ S(X,K)$ or to say better whose 0-level is $\Delta^\circ S(X,K)$ which behave appropriately with the simplicial composition.

i *The functions $\lambda(X,K)$ given by $\mathcal{A}(X,K) \to Hom_Y(X,K)_0$; $u \mapsto \widetilde{u}$ $(= X \times Y^0 \overset{\cong}{\to} X \overset{u}{\to} K)$ define a natural isomorphism.*
ii *Let $f \in Hom_Y(K,Z)_n$, $u \in \Delta^\circ S(X,K)$ and let s_0^n denote the composition*
$Hom_Y(X,K)_0 \overset{s_0}{\to} Hom_Y(X,K)_1 \overset{s_0}{\to} ...$
$\to Hom_Y(X,K)_n$
If $g \in Hom_Y(W,X)_n$ and $u \in \Delta^\circ(X,K)$ then

$$Hom_Y(W,u)_n(g) = s_0^n(u) \circ g$$

Part i is clear. For part ii notice that $s_0^n(\widetilde{u}) = u_0((S^0)^n \times 1_X)$. Furthermore if $f : Y^n \times K \to Z$ one has a composition

$$Y^n \times X \overset{\Delta(Y^n) \times 1}{\to} Y^n \times Y^n \times X \overset{1 \times u_0((s^0)^n \times 1_X)}{\to} Y^n \times K \overset{f}{\to} Z$$

On the other hand recall that

$$Hom_Y(u,Z)_n : \mathcal{A}(Y^n \times K, Z) \to \mathcal{A}(Y^n \times X, Z)$$

maps $f \mapsto (Y^n \times X \overset{1 \times u}{\to} Y^n \times K \overset{f}{\to} Z)$.

We want to prove that the composition given above and the image of f by the last function coincide. But (level wise) the image of (Y^n, X) by the composition above is

$f[1_{Y^n} \times \widetilde{u}_0 ((S^1)_n \times 1_X)](\Delta(y^n) \times 1)(y^n, x)$
$$\begin{aligned} &= f[1_{Y^n} \times \widetilde{u}_0 ((S^0)^n \times 1_X)](y^n, y^n, x) \\ &= f(y^n, \widetilde{u} ((S^0)^n(y^n)), x)) \\ &= f(y^n, \widetilde{u} (*, x)) \\ &= f(y^n, u(x)) \end{aligned}$$

Part ii is proved similarly.

Remark:

i Hom_Y with the simplicial composition given and the natural isomorphism

$$\Delta^\circ S(X,K) \to Hom_Y(X,K_0)$$

form a simplicial system which we refer to as *the simplicial system associated to Y or the Y simplicial system in $\mathcal{A}$.*

ii So far the only condition used has been that Y is pointed. For the existence of cylinder and path objects as well as for the equivalence of right and left homotopy, we will need that Y realizes products on standard models.

In order to develop path objects X^K and cylinders objects $X \otimes K$ for the models in $\mathcal{A}$, and since one of the properties desired is the equality

$$Hom_Y(X \otimes_Y K, Y) \cong (Hom_Y(X,Y))^K$$

(see in the introduction: Simplicial system), we notice that

$$Hom_Y(X,Y)^K = S_{\Delta \times \overline{K}} S_{Y \times \overline{X}}(Y)$$

Therefore it would be helpful to have a relation between composition of singular functors and singular functors of composite functors. That we do next.

Let $Y, Z : \Delta \to \mathcal{A}$ be cosimplicial objects of $\mathcal{A}$. One has the composition $\Delta \overset{Z}{\to} \Delta^\circ S \overset{R_Y}{\to} \mathcal{A}$, which is again a model in $\mathcal{A}$. Then: *There exits a natural isomorphism $\eta : S_Y \circ S_Z \to S_{R_Y \circ Z}$.*

In fact, one can see that the isomorphism of adjointness of the pair $\Delta^\circ S \overset{R_Y}{\to} \mathcal{A} \overset{S_Y}{\to} \Delta^\circ S$ applied level wise to Z gives an isomorphism

$$\eta_n : \Delta^\circ S((R_Y(Z^n), X) \leftrightarrow \Delta^\circ S(Z^n, S_Y(X))$$

with inverse, say ρ_n. It follows from naturality of the singular and realization functors that η is a simplicial map and is natural on X.

9. Existence of Cylinder and Path Objects

Let $X \in \mathcal{A}$ and $K \in \Delta^\circ S$. There exists a pair $(X \otimes_Y K, \alpha)$ with $X \otimes_Y K$ in $\mathcal{A}$ and $\alpha_K : K \to$

$Hom_Y(X \otimes_Y K)$, *which induces a natural isomorphism,*

$$\alpha_K^* : Hom_Y(X \otimes_Y K, Y) \cong [Hom_Y(X,Y)]^K$$

Remark:

The isomorphism

$$Hom_Y(X \otimes_Y K, Y) \cong [Hom_Y(X,Y)]^K$$

has not been proved to be the one used by Quillen [QD67] within the framework of the simplicial structure of a simplicial category, but it works remarkably well and we saw no reason to insist on Quillen's isomorphism. The same remark is valid in the case of path objects, next.

Let $X \in \mathcal{A}$ and $K \in \Delta^\circ S$. There exists a pair $(X^K(rel.Y), \beta)$ where $X^K(rel.Y)$ is an object of $\mathcal{A}$ and

$$\beta : K \to Hom_Y(X^K(rel.Y), X)$$

is a simplicial map, which induces a natural isomorphism

$$\beta_K^* : Hom_Y(Y, X^K(rel.Y)) \to [Hom_Y(Y,X)]^K$$

The following easy to check formulas are useful in the study of relations between the Y *simplicial system*, Y *homotopy*, and Y *pre model structures.*

Proposition:

i $\otimes_Y$ is "associative" in the following sense:

$$X \otimes_Y (K \otimes L) = (X \otimes_Y K) \otimes_Y L$$

ii $(X^K(rel.Y)^L(rel.Y) \cong X^{K \times L}(rel.Y)$.

iii $S_Y(Hom_Y(X, K)) \cong S_T(K)$ where $T = (R_Y \circ Y) \times \overline{X}$.

iv For each n, $S_Y(Hom_Y(Y,X))_n = \Delta^\circ S(Y, X^N)$ where $N = R_Y(Y^n)$.

v $S_Y(X^{R_Y(K)}) \cong (S_Y(X))^K \cong S_L(X)$ where $L = Y \times \overline{R_Y(K)}$.

10. Relative Subdivisions

In this parts we dealt with the relation among homotopies induces by models. Since there is little difference when using Δ or other category δ we use the later most of the time.

Kan [KD55,561,562] and other authors [FR68] have given characterizations of functors $\Delta^\circ S \to \Delta^\circ S$ called "subdivision functors" which are distinguished, among other things because the diagram below commutes up

to homotopy equivalence.

$| \ | : \Delta^\circ S \to Top$ denotes Milnor's geometric realization [MJ57].

Here we extend the concept of subdivision and give techniques to build subdivision functors in a more general context. First we work with the standard scheme category Δ and then, in a further generalization, any category δ is used as scheme. Thus instead of simplicial objects we work with general pre sheaves.

For the first part, instead of the geometric realization functor we use the realization of a model $M : \Delta \to Top$, and instead of normal homotopy we use that of a homotopy system γ. If $Sd : \Delta^\circ S \to \Delta^\circ S$ denotes the desired subdivision functor, $R_M : \Delta^\circ S \to Top$ the realization induced by M, and "$\overset{\sim}{}$" the homotopy equivalence induced by γ, then one would like to have that $R_M(Sd(X)) \overset{\sim}{} R_M(X)$ for each simplicial set X.

If $\mathcal{A}$ and $\mathcal{B}$ are categories, and a functor $R : \mathcal{A} \to \mathcal{B}$ admits right adjoint $S : \mathcal{B} \to \mathcal{A}$, we say that (R,S) is an adjoint pair. As for notation, $\Delta\mathcal{B}$ denotes the category of the cosimplicial objects of $\mathcal{B}$ and $\Delta^\circ\mathcal{B}$ the simplicial ones. Also, if X is a simplicial object of $\mathcal{B}$ we denote $X(n) = X_n$, and if $w : [n] \to [n]$ is an arrow of Δ, then $X(w) = w_*$. For the cosimplicial case the notation will be respectively X^n and w^*. Finally, if A is an object of $\mathcal{B}$ then the constant functor of value A will be denoted by $\ddot{A}$ when is considered as a cosimplicial object of $\mathcal{B}$ and $\underset{\cdot}{A}$ when considered as simplicial object of $\mathcal{B}$.

11. Singular Functors and Realizations

According to the theory of adjoint pairs, a functor $Y : \Delta \to \mathcal{B}$ defines a functor $S_Y : \mathcal{B} \to \Delta^\circ S$ given on the objects by $(S_Y(A))_n = \mathcal{B}(Y^n, A)$ and if $w : [n] \to [n]$ then $(S_Y(A))(w)$ is given by composition with $Y(w)$. Now, if f is a morphism of $\mathcal{B}$ then $S_Y(f)$ is given level wise by composition with f.

When $\mathcal{B}$ is a co complete category S_Y admits a left adjoint denoted $R_Y : \Delta^\circ S \to \mathcal{B}$. The categories S (sets), $\Delta^\circ S$ and Top are co complete and therefore singular functors admit realizations. We now characterize them.

Given a model $Y : \Delta \to S$ (respectively $Z : \Delta \to Top$) then its induced realization is given by

$$R_Y(X) = (\coprod_n X_n \times Y^n)/t \ ,$$

respectively,

$$R_Z(X) = (\coprod_n X_n \times Z^n)/t,$$

where $\coprod$ denotes the union (respectively the topological sum), $X_n \times Y^n$ the set theoretical product (respectively $X_n \times Z^n$ denotes the topological product taking X^n as a discreet space), and the quotient is the set theoretical (resp. topological) quotient for the equivalence relation t that for each $w : [n] \to [n]$, x in X^n, and y in Y^n (respectively z in Z^n) identifies the couples $(w_*(x), y)$ and (x, w^*, y) (respectively $(w_*(x), z)$ and (x, w^*, z)). Also, if $f : X \to K$, then $R_Y(f)([x,y]) = [f_n(x), y]$ (respectively $R_Z(f)([x,z]) = [f_n(x), z]$), where $[x,y]$ denotes the class of equivalence of the couple (x,y).

One proofs that R_Y (respectively R_Z) is a covariant functor and that is left adjoint of S_Y (respectively S_Z).

In the case of $Y : \Delta \to \Delta^\circ S$ we denote, for each Y^n in $\Delta^\circ S$, $Y^n(m) = Y^n_m$. Thus in Y^n_m, n varies contra variantly and m co variantly. Is easy to see that, for each m, Y_m defines a set theoretical model $Y_m : \Delta \to S$ given by $Y_m[n] = Y^n_m$ and for each $w : [n] \to [m]$, $Y_m(w) = (Y(w))_m$.

For each $Y : \Delta \to \Delta^\circ S$ an adjoint functor of S_Y, say R_Y, is given on the objects by $(R_Y(X))_m = R_{Y_m}(X)$, and if $r : [n] \to [m]$ then $(R_Y(X))(r) = R_{Y_r}(X)$, where $Y_r : Y_n \to Y_m$ is the cosimplicial function $(Y_r)^p = Y^p(r)$ and $R_{Y_r} : R_{Y_n} \to R_{Y_m}$ is the natural transformation induced by Y_r on the set theoretical realization functors. On the morphisms, if $f : X \to K$ then $(R_Y(f))_m = R_{Y_m}(f)$.

The proof is easy but long. A known fact, worthwhile to clarify is the following: given two adjoint pairs $\mathcal{A} \overset{R}{\to} \mathcal{B} \overset{S}{\to} \mathcal{A}$ and $\mathcal{A} \overset{R'}{\to} \mathcal{B} \overset{S'}{\to} \mathcal{A}$, there exists a one to one and onto function $G : Trans(S, S') \to Trans(R', R)$, where $Trans$ denotes the class of natural transformations of the first functor in the second. On the other hand if $Y, Y' : \Delta \to \mathcal{B}$ are two models of $\mathcal{B}$ and $a : Y \to Y'$ is a cosimplicial morphism then it induces a natural transformation $S_a : S_{Y'} \to S_Y$, given on an object A of $\mathcal{B}$ by $(S_a : (A))_n(T) = T \circ a^n$ for each T in $(S_{Y'}(A))_n$. Here we have denoted $G(a) = R_a$. In previous results this mechanism was responsible for the equality $R_{Y_r} = G(S_{Y_r})$. Notice that in the set theoretical and topological cases R_a is given by $R_a(X)[x,y] = [x, a(y)]$, while in the simplicial one is given by $(R_a : (X))_p = R_{a_p}(X)$.

The following result will be very useful in what follows:

Let $\mathcal{B}$ be a co complete category, $Y : \Delta \to \mathcal{B}$ a model on $\mathcal{B}$ and $\mathcal{B} \overset{I}{\to} \mathcal{B} \overset{J}{\to} \mathcal{B}$ an adjoint pair (notice that $\Delta \overset{Y}{\to} \mathcal{B} \overset{I}{\to} \mathcal{B}$ is a model on $\mathcal{B}$). Then there exists a natural isomorphism $P : R_{I \circ J} \cong I \circ R_Y$.

The proof follows from the fact that

$$\Delta^\circ S \overset{I \circ R_Y}{\to} \mathcal{B} \overset{S_Y \circ J}{\to} \Delta^\circ S$$

is an adjoint pair and $S_{I \circ Y}$ is naturally isomorphic to $S_Y \circ J$. In particular when $\mathcal{B}$ is S, $\Delta^\circ S$ or Top, and $I : B \to B$ denotes the functor "product by A", for a A fixed in $\mathcal{B}$, then I admits an adjoint when $\mathcal{B}$ is S and $\Delta^\circ S$, but when $\mathcal{B}$ is Top it happens only for some A. In such a case, for a model $Y : \Delta \to \mathcal{B}$ one will have that $\Delta \overset{Y}{\to} \mathcal{B} \overset{I}{\to} \mathcal{B}$ is precisely $Y \times \ddot{A}$. The proposition affirms then that

$$R_{Y \times \ddot{A}} (X) \overset{P}{\cong} R_Y(X) \times A$$

naturally.

12. Homotopies on Models

A homotopy system on a category has an extension to a system on the category of its models. Indeed if $\gamma = (I, d^0, d^1, s)$ is a homotopy system in a category $\mathcal{B}$ then γ induces on $\Delta\mathcal{B}$ the system $\Delta\gamma = (\Delta I, \Delta d^0, \Delta d^1, \Delta s)$ as follows: $\Delta I : \Delta\mathcal{B} \to \Delta\mathcal{B}$ is the normal extension of I to $\Delta\mathcal{B}$: if Y it belongs to $\Delta\mathcal{B}$ then $\Delta I(Y)^n = I(Y^n)$ and $\Delta I(Y)(w) = I(w)$. The natural transformations are given by the equalities $(\Delta d^i(Y))^n = d^i(Y^n)$ and $(\Delta s(Y))^n = s(Y^n)$, $i = 0, 1$.

Given two models Y, Z over $\mathcal{B}$ and two cosimplicial morphisms $F, G : Y \to Z$ then it is clear that $F \overset{\Delta r}{\simeq} G$ if, and only if, for each n there exists h^n (γ-homotopy) of F^n in G^n such that for each $w : [n] \to [m]$, $Z(w)h^n = h^m I(Y(w))$.

Let us see now that homotopies and homotopy equivalences on models induce homotopies and homotopy equivalences on the realizations. We first advance some observations.

Given a homotopy system γ on $\mathcal{B}$, then to the natural tranformación $d^i : 1_B \to I$ there corresponds, in a bi univocal form, a natural transformation $D^i : J \to 1_{\mathcal{B}}$, when (I, J) is an adjoint pair. Similarly, if we denote

$d^i_{R_Y} : R_Y \to I \circ R_Y$ the transformation that in X has as morphism $d^i_{R_Y} : R_Y(X) \to I(R_Y(X))$, then the corresponding transformation on the singular functors is given for each A in $\mathcal{B}$ by $S_Y(D^i(A)) : S_Y(J(A)) \to S_Y(A)$ and we denote it by $S_Y D^i$. Other two corresponding natural transformations (natural isomorphisms) that will use are $p' : S_{I \circ Y} \to S_Y \circ J$ with $p : I \circ R_Y \to R_{Y \circ I}$. Finally, there exists a cosimplicial morphism $Y \to Y \circ I$ that in the level n is $d^i_{Y_n}$. We will denote it by d^i_Y. We have then the corresponding natural transformations $R_{d^i_Y} : R_Y \to R_{I \circ Y}$ and $S_{d_{Y^i}} : S_{I \circ Y} \to S_Y$.

The following fact will be of great help in what follows: For $i = 0, 1$ the following diagram of natural transformations commutes:

$$
\begin{array}{ccc}
 & R_Y & \\
{\scriptstyle d^i_{R_Y}} \swarrow & & \searrow {\scriptstyle R_{d^i_Y}} \\
I \circ R_Y & \xrightarrow{\;p\;} & R_{I \circ Y}
\end{array}
$$

Let $\mathcal{B}$ be a category and γ a homotopy system whose cylinder I admits right adjoint J. Let Y and Z be models on $\mathcal{B}$. Then:

  i *For each pair of simplicial morphisms $F, G : Y \to Z$, if $F \overset{\Delta r}{\sim} G$ then for each simplicial set X one has that $R_F(X) \overset{\gamma}{\sim} R_G(X)$, naturally on X. In particular,*

  ii *If Y is $\Delta\gamma$-homotopically equivalent to Z, then for each simplicial set X, $R_Y(X)$ is γ-homotopically equivalent to $R_Z(X)$, naturally on X.*

This result assures that if the cylinder I of a homotopy system γ on $\mathcal{B}$ has right adjoint J, then the homotopy system induced by γ on the models, which we have denoted $\Delta\gamma$, has a direct effect on the realizations defined by the models. Indeed the proposition assures that if the models are homotopically equivalent then their realizations produce homotopically equivalent images for the original system on $\mathcal{B}$. In particular if we consider on Top the normal homotopy and if we take as Y the cosimplicial space of the topological simplexes Δ^n then R_Y is the geometric realization of Milnor. If Z is another model on Top, homotopically equivalent to the first one, then for each simplicial set X one has that $|X| \simeq R_Z(X)$, where $\simeq$ denotes the normal homotopy equivalence on Top.

Notice that although each Δ^n is homotopically equivalent to a point not all realization of simplicial sets are

so. This happens because the homotopy equivalence in question on the models is not simplicial and therefore is not an equivalence of the extended homotopy. Naturally, if a model is level wise homotopically equivalent to a point, for a given homotopy, so that the given model and the cosimplicial point are homotopically equivalent, then the relation homotopy induced by the model is trivial.

In categories with final object, an object is homotopically trivial if it is homotopically equivalent to the final object. If the concept of sub object also exists and these are preserved by morphisms, then one has that if a model is homotopically trivial then the final model object is a sub object of the given model. Said otherwise: if the final model is not a sub object of the model then the last one cannot be homotopically trivial.

In the case of S, $\Delta^\circ S$ and Top this means that if a model doesn't have cosimplicial points then it is not homotopically trivial.

This advantage, together with the one of having Eilenberg−Zilber representations make of the models without cosimplicial points specially capable tools for good homotopy theories (see for example [RCRR811]).

Up to now we have studied conditions so that two models $Y, Z : \Delta \to B$ have homotopically equivalent realizations for a homotopy system γ on $\mathcal{B}$. Now we consider functors $\Delta^\circ S \to \Delta^\circ S$ capable of preserving the homotopy type of the realizations.

Let $Y : \Delta \to \Delta^\circ S$ be a model of $\Delta^\circ S$ and $M : \Delta \to \mathcal{B}$ a model of $\mathcal{B}$. We consider a homotopy system $\gamma = (I, d^0, d^1, s)$ on $\mathcal{B}$ for which I has a right adjoint J. In general the diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\;Y\;} & {}^0 S \\
{\scriptstyle M} \searrow & & \downarrow {\scriptstyle R_M} \\
 & & \mathcal{B}
\end{array}
$$

is not commutative, but it is when Y is the model of the simplicial simplexes $\Delta[n]$. However, when the diagram commutes up to a $\Delta\gamma$-homotopy equivalence, then $R_M(S_Y(X))$ and $R_M(X)$ are γ−homotopically equivalent on $\mathcal{B}$. We develop this point next.

We will call $R_Y : \Delta^\circ S \to \Delta^\circ S$ *a subdivision* (or a subdivision of the identity) relative to the pair (γ, M) if $R_M \circ Y \overset{\Delta r}{\sim} M$. When $\boldsymbol{\Delta}$ denotes the model of the

$\Delta[n]$ simplexes, $R_{\mathbf{\Delta}}$ is the identity of $\Delta^{\circ}S$ up to isomorphism. Then what we are doing is to compare at the homotopy level R_Y with $1_{\Delta^{\circ}S}$, a notion that allows us to assimilate $R_Y(X)$ to a "subdivision" of X. For example when γ is the homotopy system induced by $M^0 \overset{d^i}{\to} M^1 \overset{s}{\to} M^0$ $(i = 0, 1)$, by means of the product, we can accept that if $Y \overset{\Delta r}{=} \mathbf{\Delta}$ then the model Y is a "subdivision" of $\mathbf{\Delta}$. It will be seen that this implies that $R_M(X) \overset{\gamma}{=} R_M(R_Y(X))$ which means that, at level of realizations (via R_M), X and $R_Y(X)$ have the same γ-homotopy type. For consistency we assimilate $R_Y(X)$ as a subdivision of X, achieved upon the subdivision Y of $\mathbf{\Delta}$ limited only by homotopy. Then one goes further using any homotopy γ, and imposing conditions on Y so that the previous results still hold. The conditions on Y and $\mathbf{\Delta}$ can hold, up to weak homotopy equivalence. That is the reason for the name of relative subdivisions.

Theorem: Let γ be a homotopy system on $\mathcal{B}$ whose cylinder admits a right adjoint. Let M be a model on $\mathcal{B}$ and Y a model on $\Delta^{\circ}S$. If R_Y is a subdivision relative to (γ, M) then there exists a natural transformation $a : R_M \to R_M \circ R_Y$ such that for each X in $\Delta^{\circ}S$, $a_X : R_M(X) \to R_M(R_Y(X))$ is a γ-homotopy equivalence.

For the proof, let us remember that, since $R_M \circ Y \overset{\Delta r}{=} M$, there exists a transformation $e : R_{R_M \circ Y} \to R_M$ such that for each X, e_X is a γ-homotopy equivalence. The theorem will be proved if we exhibit a natural isomorphism $R_M \circ R_Y \to R_{R_M} \circ Y$, or equivalently one natural isomorphism $S_{R_M \circ Y} \to S_Y \circ S_M$, but it is obtained by means of the following chain of natural isomorphisms $(S_{R_M \circ Y}(A))_n = \mathcal{B}((R_M \circ Y)^n, A) = \mathcal{B}(R_M(Y^n), A) = \Delta^{\circ}S(Y^n, S_M(A)) = (S_Y(S_M(A)))_n$.

A pair (Y, M), where Y is a model of $\Delta^{\circ}S$ and M is a model of $\mathcal{B}$, is said to be a *singular pair* if $R_M \circ Y \cong M$.

Is clear that if (Y, M) is a singular pair then there exists a natural isomorphism $R_M \circ R_Y \to R_M$. Therefore Y is a subdivision relative to (γ, M) for any homotopy system γ on $\mathcal{B}$. The converse is also true since "=" is the homotopy relation of particular homotopy systems on any category. For example that is the case for any homotopy system in which $d^0 = d^1$.

Up to now we have defined subdivisions of the identity relative to pair (γ, M). Now we consider subdivisions of any models.

Let γ be a homotopy system on $\mathcal{B}$ whose cylinder admits a right adjoint and let M be a model of $\mathcal{B}$. If Y

and Z are models of $\Delta^{\circ}S$ we say that Y *is a subdivision of Z relative to* (γ, M), denoted $Y \overset{(\gamma, M)}{\sim} Z$, if $R_M \circ Y \overset{\Delta r}{\sim} R_M \circ Z$. Notice that if the homotopy of γ is transitive then the relation $\overset{(\gamma, M)}{\sim}$ is an equivalence relation on the class of the models of $\Delta^{\circ}S$.

The relation $\overset{(\gamma, M)}{\sim}$ compares models through its realizations in a weak way, via the homotopy equivalence induced by γ on the models of $\mathcal{B}$. Also, it compares their realizations (via R_M) by means of the homotopy equivalence of γ in $\mathcal{B}$. For if Y, Z, γ, and M are as above and if $Y \overset{(\gamma, M)}{\sim} Z$ then there exists a natural transformation $a : R_M \circ R_Y \to R_M \circ R_Z$ such that for each X in $\Delta^{\circ}S$ the morphism $a_X : R_M(X) \to R_M(R_Z(M))$ is a γ-homotopy equivalence.

13. Subdivisions on Pre Sheaves

It is known that a covariant functor $Y : \delta \to \mathcal{A}$, where δ and $\mathcal{A}$ are any categories, induces a covariant functor $S_Y : \mathcal{A} \to \delta^{\circ}S$. If $\mathcal{A}$ is co complete then S_Y has a left adjoint functor, denoted here by $R_Y : \delta^{\circ}S \to \mathcal{A}$. Generalizing the terminology of [MJ57] and [RR76] we call

i The functor $Y : \delta \to \mathcal{A}$ a *δ-model of $\mathcal{A}$*, or a *model of $\mathcal{A}$* when δ is clear.

ii S_Y the *singular functor* and R_Y the *realization functor* of Y.

So far we have studied the repercussions on the functors R_Y and R_Z produced by the existence of a homotopy equivalence $Y \simeq Z$, induced in $\delta \mathcal{A}$ by a preset homotopy of $\mathcal{A}$ when δ is the category Δ. The homotopy induced in $\Delta \mathcal{A}$ imposes conditions of naturality that occasionally can be very restrictive. One can see however that if one considers naturality on a restricted number of arrows of Δ, then the theory still works. This is equivalent to change the category Δ restricting arrows.

For generalization we take any unrestricted category δ and work as follows: first we give the extension of a homotopy system of $\mathcal{A}$ to $\delta \mathcal{A}$ ($\delta \mathcal{A}$ the category of covariant functors $\delta \to \mathcal{A}$ and natural transformations as morphisms) and we show that if Y and Z are models of $\mathcal{A}$, homotopically equivalent for that extension, then for each pre sheaf X of $\delta^{\circ}S$, it happens that in $\mathcal{A}$, $R_Y(X)$ is homotopically equivalent to $R_Z(X)$, naturally on X. Further we fix a realization $R_M : \delta^{\circ}S \to \mathcal{A}$ and generalize the case of [KD55,561,562] to give conditions on two functors $F, G : \delta^{\circ}S \to \delta^{\circ}S$ so that the realizations

of $F(X)$ and $G(X)$ for R_M are naturally homotopically equivalent. Those said conditions define a relationship on the functors $\delta^\circ S \to \delta^\circ S$ that will be called of *relative subdivision*, extending the terminology of [KD57]. In general a standard homotopy in $\delta^\circ S$ doesn't exist. So we fix a homotopy system there, another in $\mathcal{A}$, and we give conditions relating these two systems by means of the functor $R_Y : \delta^\circ S \to \mathcal{A}$ induced by $Y : \delta \to \mathcal{A}$ and we say that "Y carries the system of $\delta^\circ S$ into the one of $\mathcal{A}$".

If both Y and Z carry the system of $\delta^\circ S$ into the one of $\mathcal{A}$ and if Y is homotopically equivalent to Z (for the homotopy of $\delta^\circ S$ extended from $\mathcal{A}$) then for each A of $\mathcal{A}$, $S_Y(A)$ is homotopically equivalent to $S_Z(A)$ in $\delta^\circ S$. When $\delta = \Delta$ and homotopy is the normal one in $\Delta^\circ S$ and $Y \simeq Z$ then the homologies induced by Y and Z in $\mathcal{A}$ are isomorphic.

With normal changes the proofs for Δ can be adapted to the new generalization.

In what follows $\mathcal{A}$ denotes a co complete category. For two models $Y, Z : \delta \to \mathcal{A}$ and a morphism of models $f : Y \to Z$ (natural transformation) we denote by $S_f : S_Z \to S_Y$ the transformation induced by f on singular functors.

We dealt now with the construction of a natural transformation $R_f : R_y \to R_z$.

Recall by [RCRR812] that if we denote by $\Phi_Y : 1 \to S_Y \circ R_Y$ and $\Psi_Y : R_Y \circ S_Y \to 1$ adjointness transformations for (R_Y, S_Y), by Ψ_Z, Φ_Z, those of (R_Z, S_Z) and by

$$\begin{array}{ccc} \mathcal{A}(R_Y(X), A) & \longleftrightarrow & \delta^\circ S(X, S_Y(A)) \\ \alpha & \to & \alpha^Y \\ \beta_Y & \leftarrow & \beta \end{array}$$

(respectively $\alpha \to \alpha^Z$, $\beta \to \beta_Z$) the adjointness isomorphisms, then

$$R_f(X) = [S_f(R_Z(X)) \circ \Phi_Z(X)]_Y : R_Y(X) \to R_Z(X)$$

The extension of homotopy systems on $\mathcal{A}$ to systems on $\delta\mathcal{A}$ is the following one: If $\eta = (I, d^0, d^1, s)$ is a homotopy system on $\mathcal{A}$ then $\delta\eta = (\delta I, \delta d^0, \delta d^1, \delta s)$ denotes the system in $\delta\mathcal{A}$ with cylinder $\delta I(Y) = I \circ Y$ and $\delta I(\lambda)_x = I(\lambda_x)$ for each morphism $\lambda : Y \to Z$ in $\delta\mathcal{A}$, where $\delta d^i(Y) : Y \to \delta I(Y)$ is given by $\delta d^i(Y)_x = d^i(Y(x))$ and $\delta s(Y)$ by $\delta s(Y)_x = s(Y(x))$, $x \in \delta$.

The δn homotopy is a natural extension of the homotopy η because if Y, Z are objects of $\delta\mathcal{A}$ and $F, G : Y \to Z$ morphisms, then $F \overset{\delta\eta}{\sim} G$ (F is homotopic to G via $\delta\eta$) if, and only if, there exists a family

$h_X : I(Y(X)) \to Z(X)$ *of η-homotopies $F_X \sim\to G_X$ such that if $w : X \to X'$ is a morphism of δ, then* $Z(w)_0 h_X = h_{X'} \circ I(Y(x))$.

Since the homotopy relation of a homotopy system is not generally an equivalence relation we fix and keep an order that we exemplify for the system $\delta\eta$: If Z, Y are two models of $\mathcal{A}$ (on δ) then Z is $\delta\eta$ homotopically equivalent to Y if there exist morphisms $F : Z \to Y$ and $G : Y \to Z$ such that $G \circ F \overset{\delta\eta}{\sim} 1_Z$ and $F \circ G \overset{\delta\eta}{\sim} 1_Y$.

If the cylinder of $\eta = (I, d^0, d^1, s)$ admits right adjoint $J : \mathcal{A} \to \mathcal{A}$ then J is completed in a right homotopy system denoted $\eta^* = (J, D^0, D^1, S)$, for which $f \overset{\eta}{\sim} g$ if and only if $f \overset{\eta^*}{\sim} g$. In fact if we denote the adjointness transformations by $\Phi : 1_{\mathcal{A}} \to J \circ I$ and $\Psi : I \circ J \to 1_{\mathcal{A}}$. For $\alpha : I(A) \to B$, $\alpha^* = J(\alpha) \circ \Phi_A$ and for $\beta : A \to J(B)$, $\beta_* = \Psi_B \circ I(\beta)$, then to the transformations d^i $(i = 0, 1)$ there correspond the transformations $D_A^i : J \to 1_A$ given for each A in $\mathcal{A}$ by $D_A^i = \Psi_A \circ d^i J(A)$ or $\overline{d^i} = D^i$ with the notation of [RCRR811].

For a model $Y : \delta \to \mathcal{A}$, one has another model $\delta \overset{Y}{\to} \mathcal{A} \overset{I}{\to} \mathcal{A}$ on $\mathcal{A}$ whose realization functors $R_{I \circ Y}$ and singular $S_{I \circ Y}$ are related with R_Y and S_Y in the following way.

The adjoint pairs $(I \circ R_Y, S_Y \circ J)$ and $(R_{I \circ Y}, S_{I \circ Y})$ are equivalent. That is to say that there exists a natural isomorphism $\overline{\rho_Y} : S_{I \circ Y} \to S_Y \circ J$ (and, equivalently, a natural isomorphism $\rho_Y : I \circ R_Y \to R_{I \circ Y}$).
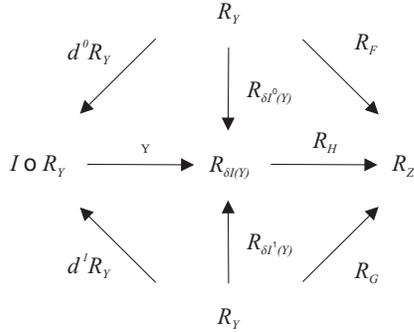
In fact $(\overline{\rho}_Y(A))_X$ is the isomorphism of $\mathcal{A}$ given by the following chain of natural isomorphisms: $[S_{I \circ Y}(A)]_X = \mathcal{A}(I \circ Y_{(X)}, A) = \mathcal{A}(I(Y_{(X)}), A) \overset{*}{=} \mathcal{A}(I \circ Y_{(X)}, A) = S_Y(J(A))_X$, where ρ_Y is the isomorphism induced by $\overline{\rho}_y$ at realizations.

The effect on the realizations, of homotopy among models, is the following one:

Let $Y, Z : \delta \to \mathcal{A}$ be two models and $F, G : Y \to Z$ morphisms of $\mathcal{A}$. If $F \overset{\delta\eta}{\sim} G$ then there exists a natural transformation $\mathcal{H} : I \circ R_Y \to R_Z$ such that for each X in $\delta^\circ S$, $\mathcal{H}_X$ defines a homotopy $R_F(X) \overset{\mathcal{H}_X}{\sim} R_G(X)$.

That is so because a $\delta\eta$ homotopy of F into G, say H produces a commutative diagram, where of course $d^i R_Y$ $(i = 0, 1)$ is the transformation given by $d^i R_Y(X) = d^i(R_Y(X))$. One takes $\mathcal{H} = R_H \circ \rho_Y$. As a consequence one also has that *if Y and Z are models of $\mathcal{A}$ and Y is $\delta\eta$ homotopically equivalent to Z then there exists a natural transformation $\lambda : R_Y \to R_Z$ such that for*

each X in $\delta^\circ S$, $\lambda_X : R_Y(X) \to R_Z(X)$ is a η homotopy equivalence.

$$R_Y$$

$$d^0 R_Y \quad\quad R_F$$

$$R_{\delta I^0(Y)}$$

$$I \circ R_Y \xrightarrow{\ \ \ Y\ \ \ } R_{\delta I(Y)} \xrightarrow{\ \ R_H\ \ } R_Z$$

$$d^1 R_Y \quad\quad R_{\delta I^1(Y)} \quad\quad R_G$$

$$R_Y$$

When $\mathcal{A} = Top$ (or Kelly) and $\delta = \Delta$ for a model $Y : \Delta \to Top$ the realizations $R_Y(X)$ will be called $Y - CW$ complexes. They are thought of as complexes with (in general non Euclidean) cells Y^n. The last result establishes a sufficient condition so that the $Y - CW$ complex $R_Y(X)$ is homotopically equivalent to the CW complexes $|X|$ ($|\ |$ Milnor's realization of [MJ57]).

In the topological case $Y - CW$ complexes are a particular case of cellular complexes with non Euclidean cells [RCRR812] studied by the author, professor Carlos Ruiz at National University of Colombia, and Joaquin Luna [RCLJ82]. However, for any co complete category $\mathcal{A}$ and any model $Y : \delta \to \mathcal{A}$ on $\mathcal{A}$ the concept of $Y -$complex exist. In fact the sub category of $Y -$complexes of $\mathcal{A}$ is precisely $R_Y(\delta^\circ Conj)$, $R_Y : \delta^\circ Conj \to \mathcal{A}$.

Moving to the case of relative subdivisions among functors $\delta^\circ S \to \delta^\circ S$ we think of an object of $\delta^\circ S$ as a set theoretical skeleton that serves as a pattern to patch the pieces provided by $Y(X)$ of $\mathcal{A}$ to obtain an object of $\mathcal{A}$. Up to now we have "modified" a model Y to obtain another Z, in such a way that for each skeleton X the obtained objects, one via patching objects $Y(X)$ and the other via patching objects $Z(X)$, were homotopically equivalent. Now we are interested in how to modify the skeleton X into another X' so that for $Y : \delta \to \mathcal{A}$ fixed, the objects obtained of them, patching $Y(X)$ and $Y(X')$, are homotopically equivalent in $\mathcal{A}$.

We consider the process $X \mapsto X'$ as a functor $\delta^\circ S \to \delta^\circ S$ which is the result of a "modification" of the (identity) functor $X \mapsto X$. Accepting that, it is clear that a more general situation arises considering two functors $F, G : \delta^\circ S \to \delta^\circ S$ for which we want to decide in which sense $G(X)$ is a "modification" of $F(X)$, in such a way that $G(X)$ and $F(X)$ are homotopically equivalent.

We use the way suggested by the precedent theory namely, we consider the realization functors $\delta^\circ S \xrightarrow{F} \delta^\circ S \xrightarrow{R_Y} \mathcal{A}$ and $\delta^\circ S \xrightarrow{G} \delta^\circ S \xrightarrow{R_Y} \mathcal{A}$, provided F and G each admits right adjoint. In such a case F and G are "modelable". In other words they are realizations induced by models and we can concentrate on the models that define them. We will use the term "subdivision" rather than "modification" following the case exposed by Kan [KD57].

Let us fix a model $M : \delta \to \mathcal{A}$ and a system η in $\mathcal{A}$ with right adjoint η^*. Let $Y, Z : \delta \to \delta^\circ S$ be models of $\delta^\circ S$. We will say that Z *is a subdivision of Y relative to the couple* (η, M) if $R_M \circ Y$ is $\delta\eta$ homotopically equivalent to $R_M \circ Z$. We denote it by $Y \overset{(\eta, M)}{\hookrightarrow} Z$.

If the homotopy η is transitive then $\overset{(\eta, M)}{\hookrightarrow}$ is an equivalence relation on the models of $\delta^\circ S$. With no conditions on η one has the wanted result: *If $Y \overset{(\eta, M)}{\hookrightarrow} Z$, then there exists a natural transformation $L : R_M \circ R_Y \to R_M \circ R_Z$ such that for each object of $\delta^\circ S$, $L_X : R_M(R_Y(X)) \to R_M(R_Z(X))$ is a η homotopy equivalence.*

When F and G admit right adjoint it will be said that G *is a subdivision of F relative to* (η, M) if $F \circ \boldsymbol{\delta} \overset{(\eta, M)}{\hookrightarrow} G \circ \boldsymbol{\delta}$ ($\boldsymbol{\delta} : \delta \to \delta^\circ S$ the natural inclusion). Since $F \circ \boldsymbol{\delta}$ (respectively $G \circ \delta$) is the model defining F (respectively G), what we have is that if G is a subdivision of F relative to (η, M) then for each X in $\delta^\circ S$, $R_M(F(X))$ is homotopically equivalent to $R_M(G(X))$ naturally on X. If $G \overset{(\eta, M)}{\hookrightarrow} 1_{\delta \circ S}$ we will simply say that G is a "subdivision functor module (η, M)".

Notice that so far we have obviated the use of homotopy systems in $\delta^\circ S$. However there exists an intermediate step giving a homotopy system in $\delta^\circ S$ that is carried by M into η (see below). If Y is homotopically equivalent to Z for the extension of the homotopy from $\delta^\circ S$ then $Y \overset{(\eta, M)}{\hookrightarrow} Z$. In such a case, a sufficient condition for $R_Z(\delta^\circ S \to \delta^\circ S)$ to be a subdivision of R_Y relative to (η, M) is that $Y \simeq Z (mod\ \delta n)$.

14. Isomorphic Homologies

Returning to the case $\Delta \to \mathcal{A}$, the singular functor $S_Y : \mathcal{A} \to \Delta^\circ S$ gives place in an obvious way to a homology on $\mathcal{A}$ induced by Y. With the same homotopy concept among models, $Y, Z : \Delta \to \mathcal{A}$ one can compare S_Y with S_Z using the standard homotopy of $\Delta^\circ S$. The process is also generalizable as we will see later. In this

paragraph we consider the effect produced by homotopy equivalencies of models on the respective singular functors.

We consider in $\mathcal{A}$ a homotopy system $\eta = (I, d^0, d^1, s)$, with right adjoint $\eta^* = (J, D^0, D^1, \mathcal{S})$. There are then adjoint pairs $\mathcal{A} \xrightarrow{I} \mathcal{A} \xrightarrow{J} \mathcal{A}$ and $\mathcal{A} \xrightarrow{1_{\mathcal{A}}} \mathcal{A} \xrightarrow{1_{\mathcal{A}}} \mathcal{A}$. Once fixed adjointness transformations there exist isomorphisms $Trans(1_{\mathcal{A}}, I) \to Trans(J, 1_{\mathcal{A}})$ and $Trans(I, 1_{\mathcal{A}}) \to Trans(1_{\mathcal{A}}, J)$. The first one sends d^i in D^i ($i = 0, 1$) and the second sends s in $\mathcal{S}$. If one also has an adjoint pair $\mathcal{A} \xrightarrow{R} \mathcal{B} \xrightarrow{S} \mathcal{A}$, with fixed adjointness transformations then there are two adjoint pairs in which we are interested namely $\mathcal{A} \xrightarrow{R} \mathcal{B} \xrightarrow{S} \mathcal{A}$ and $\mathcal{A} \xrightarrow{RI} \mathcal{B} \xrightarrow{JS} \mathcal{A}$, together with the following natural transformations and notations:

- $R(d^i_A) : R(A) \to R(I(A))$, $A \in \mathcal{A}$, $i = 0, 1$, denoted Rd^i.
- $R(s_A) : R(I(A)) \to R(A)$, $A \in \mathcal{A}$, denoted Rs.
- $D^i_{S(B)} : J(S(B)) \to S(B)$, $B \in \mathcal{B}$, $i = 0, 1$, denoted D^i_S.
- $S_{\mathcal{S}(B)} : S(B) \to J(\mathcal{S}(B))$, $B \in \mathcal{B}$, denoted $S_{\mathcal{S}}$.

One can verify that if the adjointness transformations of (RI, JS) are those obtained from the pairs (I, J) and (R, S), the isomorphism $Trans(R, RI) \to Trans(JS, S)$ then sends Rd^i in D^i_S ($i = 0, 1$) and $Trans(RI, R) \to Trans(S, JS)$ sends R_S in $S_{\mathcal{S}}$. Similarly if $\beta \xrightarrow{H} \mathcal{A} \xrightarrow{T} \mathcal{B}$ is an adjoint pair then also is $\mathcal{B} \xrightarrow{H} \mathcal{A} \xrightarrow{I} \mathcal{A} \xrightarrow{J} \mathcal{A} \xrightarrow{T} \mathcal{B}$ and we have transformations and notations as follows:

- $d^i_{H(B)} : I(H(B)) \to H(B)$, $B \in \mathcal{B}$, $i = 0, 1$ denoted d^i_H.
- $\mathcal{S}_{H(B)} : H(B) \to I(H(B))$, $B \in \mathcal{B}$, denoted $\mathcal{S}_H$.
- $T(D^i_A) : T(J(A)) \to T(A)$, $A \in \mathcal{A}$, $i = 0, 1$ denoted TD^i.
- $T(\mathcal{S}_A) : T(A) \to T(J(A))$, $A \in \mathcal{A}$, denoted $T\mathcal{S}$.

with the obvious correspondences among them. Using the notation of [RCRR811] for two adjoint pairs $\mathcal{A} \xrightarrow{F_i} \mathcal{B} \xrightarrow{G_i} \mathcal{A}$ ($i = 1, 2$) that assigns $r \to \bar{r}$ by the isomorphism $Trans(F_1, F_2) \to Trans(G_1, G_2)$ one has the following group of formulas:

i	$\overline{Rd^i} = D^i_S$.	iii $\overline{d^i_H} = TD^i$.
ii	$\overline{R_S} = S_{\mathcal{S}}$.	iv $\overline{s_H} = T\mathcal{S}$.

In our case we are using models $Y : \delta \to \mathcal{A}$ each one with a adjoint pair $\delta^{\circ}S \xrightarrow{R_Y} \mathcal{A} \xrightarrow{S_Y} \delta^{\circ}S$, a homotopy system $\boldsymbol{\eta} = (\boldsymbol{I}, \boldsymbol{d}^0, \boldsymbol{d}^1, \boldsymbol{s})$ in $\delta^{\circ}S$ with right adjoint

$\boldsymbol{\eta}^* = (\mathbb{J}, \mathbb{D}^0, \mathbb{D}^1, \mathbb{S})$ and the system η in $\mathcal{A}$ with adjoint η^*.

We will say that Y *realizes* $\boldsymbol{\eta}$ *into* η (or equivalently $\boldsymbol{\eta}^*$ realizes into η^*) if a natural transformation $\epsilon : R_Y \boldsymbol{I} \to I R_Y$ exists such that $\epsilon \circ (R_Y \boldsymbol{d}^i) = d^i_{R_Y}$ for $i = 0, 1$.

As has been seen in [RCRR811], if $\boldsymbol{\mathcal{I}} : \delta \to \delta^{\circ}S$ denotes the inclusion functor, then the previous definition can be given equivalently, restricting $I R_Y$ and $R_Y \boldsymbol{I}$ to $\boldsymbol{\mathcal{I}}(\delta)$. That is to say demanding that for each x of δ a morphism $\epsilon_{Y(x)} : I(Y(x)) \to R_Y(\boldsymbol{I}(Y(x)))$ exists such that $\epsilon_{Y(x)} \circ (R_Y \boldsymbol{d}^i)_{Y(x)} = (d^i_{R_Y})_{Y(x)}$ naturally on x.

When one uses $\Delta^{\circ}S \xrightarrow{R_Y} \mathcal{A} \xrightarrow{S_Y} \Delta^{\circ}S$ with the normal system in $\Delta^{\circ}S$, Y^0 final object of $\mathcal{A}$ and η the system induced by product with $Y^0 \xrightarrow{d_i} Y^1$ ($i = 0, 1$) one has a natural transformation, $\epsilon_K : R_Y(K \times \Delta[1]) \to R_Y(K) \times R_Y(\Delta[1]) \simeq R_Y(K) \times Y^1$, the first part of which is induced by the projections and the second by the natural isomorphism $R_Y(\Delta[n]) \simeq Y^n$ connecting the normal cylinder in $\Delta^{\circ}S$ with the one of $R_Y(K)$ in $\mathcal{A}$. One also has the following commutative diagram for the inclusion $d^i_K : K \to K \times \Delta[1]$.

$$
\begin{array}{ccc}
& R_Y(K) & \\
{\scriptstyle R_Y(\boldsymbol{d}^i_K)} \swarrow & & \searrow {\scriptstyle \boldsymbol{d}_{R_Y(K)}} \\
R_Y(K \times \Delta[1]) & \xrightarrow{\quad \kappa \quad} & R_Z(K) \times Y^1
\end{array}
$$

It shows that R_Y also connects the inclusions of objects into their cylinders. This is the situation we just generalized. In [RCRR811] conditions were given so that ε becomes an isomorphism, case in which R_Y respects cylinders together with the inclusions and therefore transmits homotopies. Our condition here is then weaker than the one used normally for transmission of homotopies.

Let us suppose now that $F, G : Y \to Z$ are morphisms between models $Y, Z : \delta \to \mathcal{A}$. For each A, object of $\mathcal{A}$, one has two morphisms $S_F(A), S_G(A) : S_Z(A) \to S_Y(A)$. Let us see the implication derived on them by the fact that F and G are homotopic via $\delta\eta$.

Let us suppose that Y realizes $\boldsymbol{\eta}$ into η. Let $F, G : Y \to Z$ be two morphisms. If F is $\delta\eta$-homotopic to G, then a natural transformation $S_Z(A) \to \mathbb{J}(S_Y(A))$ exists that is a right (or $\eta^* -$) homotopy of $S_F(A)$ into $S_G(A)$, for each A.

The proof is long and will be omitted.

Let $Y, Z : \Delta \to \mathcal{A}$ be two models of $\mathcal{A}$ which realize $\boldsymbol{\eta}$ into η. If Y is $\delta\eta$ homotopically equivalent to Z then a natural transformation $\lambda : S_Y \to S_Z$ exists such that λ_A is a $\boldsymbol{\eta}$ (equivalently $\boldsymbol{\eta}^*$) homotopy equivalence.

15. Examples

Let $Y : \Delta \to \mathcal{A}$ be a cosimplicial model of $\mathcal{A}$ where $\mathcal{A}$ is co complete, closed for finite products and with final object $*$. Let us suppose that $Y^0 = *$. Then one has in $\mathcal{A}$ a homotopy system $\eta = (I, d^0, d^1, s)$ where

$$I(A) = A \times Y^1$$
$$d^i_A : A \simeq A \times Y^0 \stackrel{1 \times d^i}{\to} A \times Y^1 (d^i : Y^0 \to Y^1)$$
$$s_A : A \times Y^1 \stackrel{1 \times s^0}{\to} A \times Y^0 \cong A.$$

Let us suppose now that $(-) \times Y_1$ has right adjoint. It is usually denoted $(\ )^{Y^1}$. If $\boldsymbol{\eta}$ denotes the normal system in $\Delta^\circ S$ then Y realizes $\boldsymbol{\eta}$ in η. If a priori Y is $\Delta\eta$ equivalent to Z then a cosimplicial morphism exists $F : Z \to Y$, equivalence of $\Delta\eta$ homotopy, for which $d^i \circ F^0 = F^1 \circ d^i$.

This in turn implies at long last the following commutative diagram

Where Pr denotes the map $(R_Z(\pi_1), iso \circ R_Z(\pi_2))$. Therefore one has the following:

Let $\mathcal{A}$ be a co complete category, closed for finite products, with final object $*$, and let $Y : \Delta \to \mathcal{A}$ be such that $Y^0 = *$ and $(-) \times Y^1$ has right adjoint. Let η be the system induced by $* = Y^0 \stackrel{d^i}{\to} Y^1 \stackrel{s^0}{\to} Y^0$ $(i = 0, 1)$, and $Z : \Delta \to \mathcal{A}$ any model of $\mathcal{A}$. If Y is $\Delta\eta$ homotopically equivalent to Z then a natural transformation $\lambda : S_Y \to S_Z$ exists such that for each A of $\mathcal{A}$, $\lambda_A : S_Y(A) \to S_Z(A)$ is a homotopy equivalence in $\Delta^\circ S$.

Now we consider examples of homotopically equivalent models. There is a class of models in Top for which the homotopy extended from Top preserve details from the original that can be helpful. As an illustration let Y be a model of Top. We say that Y is convex if for each

n, Y^n is a convex subspace of a vector topological space (on $\mathbb{R}$) and for each $w : [n] \to [m]$, $w^* : Y^n \to Y^m$ is a lineal function. Then

   i If Y is a convex model, Z is any model of Top and $F, G : Z \to Y$ are any two cosimplicial continuous functions, then F is homotopic to G by the homotopy of ΔTop, extension of the normal homotopy of Top.

   ii If Y and Z are convex models of Top admitting cosimplicial functions $Y \to Z$ and $Z \to Y$ then Y is homotopically equivalent to Z for the homotopy of ΔTop extension of that of Top.

In what follows the homotopy of ΔTop (respectively of ΔKelly) extension of the normal homotopy of Top (respectively of Kelly) will be called the **normal homotopy of $\boldsymbol{\Delta Top}$** (respectively of ΔKelly).

We have that if Y is a convex model of Top (respectively Kelly) with cosimplicial points, then Y is null-homotopic, that is to say, homotopically equivalent (for the normal homotopy) to a cosimplicial point.

The converse is true for any not necessarily convex space. In fact, if Y has cosimplicial points and Z doesn't have, then Y and Z are not homotopically equivalent.

For the model of the complexes Δ^n of Top (or Kelly) its realization is the geometric realization [MJ57]. It is clear that $|K|$ is not in general equivalent (homotopically) to the discreet space $|_-| \circ K$, which happens to be the realization $R_Y(K)$ for Y a cosimplicial point, even thought each Δ^n is homotopically null.

Our definition of homotopy among models is somehow taking in consideration these facts. For convex spaces of the category Kelly the relationship is very precise as we will see. But for the general case the existence of a natural transformation $\lambda : R_Y \to R_Z$ such that for each X in $\Delta^\circ S$, λ_X is a η homotopy equivalence doesn't seem to imply that $Y \stackrel{\delta\eta}{\simeq} Z$, unless λ_X^{-1} form a natural transformation at least for $X = \Delta[n]$, $n = 0, 1, \cdots$.

Recall that for $Y : \Delta \to \mathcal{A}$, $RC(Y)$ ("The right cut of Y", see [RR76]) is the cosimplicial object of $\mathcal{A}$ defined this way: $[RC(Y)]^n = Y^{n+1}$, the $d^i : RC(Y)^n \to RC(Y)^{n+1}$ are the same $d^i : Y^{n+1} \to Y^{n+2}$ but only for $i = 0, 1, ..., n + 1$. Similarly for the $s^j : RC(Y)^n \to RC(Y)^{n-1}$ are the same $s^j : Y^{n+1} \to Y^n$ but only for $j = 0, ...n - 1$. It is easy to see that $RC : \Delta\mathcal{A} \to \Delta\mathcal{A}$ is a covariant functor and that $d : 1 \to RC$ given for

$[d(Y)]^n = d^{n+1} : Y^n \to Y^{n+1}$ is a natural transformation. If $Y : \Delta \to Top$ (respectively Kelly) is convex then so is $RC(Y)$. Therefore if $Y : \Delta \to Top$ is a convex model and there exists a continuous cosimplicial function $RC(Y) \to Y$ then Y and $RC(Y)$ are homotopically equivalent for the normal homotopy in ΔTop.

Taking the case of Δ one has that:

$$d^i : RC(\Delta)^n \to RC(\Delta)^{n+1}$$
$$(x_0, ..., x_n) \mapsto (x_0, ..., x_{i-1}, x_i, ..., x_{n+1})$$

for $i = 0, ..., n+1$

$$s^j : RC(\Delta)^n \to RC(\Delta)^{n+1}$$
$$(x_0, ..., x_{n+1}) \mapsto (x_0, ..., x_j + x_{j+1}, ..., x_{n+1})$$

for $j = 0, ..., n+1$

$RC(\Delta)$ is not homotopically equivalent to Δ because the first one has the cosimplicial point $P^n = (0, ..., 0, 1) \in \mathbb{R}^{n+2}$.

That example shows that, in general, do not exist simplicial morphisms $RC(Y) \to (Y)$ and also that $RC(\Delta)$ is homotopically null.

Now $RC(\Delta) - P$ is a cosimplicial space and $d : \Delta \to RC(\Delta) - P$ since $P^n \notin d^n(\Delta^n)$. Further $RC(\Delta) - P$ is convex since $(0, ..., 0, 1)$ is a vertex of Δ^{n+1}. Finally the $f^{n-1} : (\Delta^n - P^{n-1}) \to \Delta^{n-1}$ that maps $(x_0, ..., x_n) \mapsto \left(\sum_{i=0}^{n-1} x_i\right)^{-1} (x_0, ..., x_{n-1})$ is a cosimplicial continuous function $(RC(\Delta) - P) \to (\Delta - P^n)$.

Therefore one can also "extend" Δ to a model W with $W^n = R^{n+1}$ co faces and co degeneracies defined identically as those of Δ, thus Δ forming a cosimplicial subspace of W. But W also has a cosimplicial point, namely 0. Therefore it is homotopically null and cannot exist continuous cosimplicial functions $W \to \Delta$. Contrary to the previous case $W - 0$ is not a cosimplicial space. But if we consider $\mathcal{H}^n = \{(x_0, ..., x_n) | \sum_{i=0}^{n} x_i = 0\}$ then $W - \mathcal{H}$ is a cosimplicial space and $0 \subseteq \mathcal{H} \subseteq W$. Now the function $W - \mathcal{H} \to \Delta$, $(x_0, ..., x_n) \mapsto \left(\sum_{i=0}^{n} x_i\right)^{-1} (x_0, ..., x_n)$ is continuous cosimplicial. However we cannot apply results for convex models since $W - \mathcal{H}$ is not convex (it is not connected). In this case it is clear that $W - \mathcal{H}$ is not homotopically equivalent to Δ since for each $n \neq 0$, $(W - \mathcal{H})^n$ is not homotopically null while Δ^n is, and clearly from the definition one has

that a necessary condition for $Y \overset{\delta\eta}{\simeq} Z$ is that for each $x \in \delta$, $Y(x) \overset{\eta}{\simeq} Z(x)$.

The model $\mathcal{H}$ will be called by obvious reasons the **model of the hyperplane of W**. We notice that $\mathcal{H}^0 = \{0\}$ and $\mathcal{H}^1$ is the straight line at $l35°$ that goes trough the origin. Also $d^0 = d^1 : \mathcal{H}^0 \to \mathcal{H}^1$, which implies the interesting fact that this model's homotopy is the equality and of course, homotopy equivalencies are the homeomorphisms.

If we consider $\mathcal{H}(k)^n = \{(x_0, ..., x_n) \in R^{n+1} \mid \sum x_i = k\}$ with co faces and co degeneracies like the ones of W, then for each n, $\mathcal{H}(k)^n$ is homeomorphic to $\mathcal{H}^n$. However $\mathcal{H}(k)$ doesn't have cosimplicial points if $k \neq 0$. This can be seen from the fact that if $\mathcal{H}(k)^0$ has cosimplicial points then has only one, since $\mathcal{H}(k)^0$ has one element, which most belong to the cosimplicial point. But $d^0(k) = (0, k)$ and $d^1(k) = (0, k)$. As $(0, k) = (k, 0) \leftrightarrow k = 0$, then $\mathcal{H}(k)$ doesn't have cosimplicial points if $k \neq 0$, although $R_{\mathcal{H}}$ and $R_{\mathcal{H}}(k)$ are level wise homeomorphic. Since $\partial\Delta[1]$ is realized by $R_{\mathcal{H}}$ into $\{(0,0)\} \subseteq \{(x, y) \in R^2 \mid x + y = 0\}$ and by $R_{\mathcal{H}}(k)$ into the subspace $\{(0, k), (k, 0)\}$ of $\{(x, y) \in R^2 \mid x + y = k\}$ then it is clear that the realizations are not homotopically equivalent. Apart from this $\mathcal{H}(k) \simeq \mathcal{H}(l)$ if $k \neq 0 \neq l$.

The question on whether two models can have homotopically equivalent realizations without being themselves homotopically equivalent is partially clarified for convex models of the category Kelly with next result which improves the late version: Let Y, Z be two convex models of Kelly. The following statements are equivalent:

   i Y is homotopically equivalent to Z for the normal homotopy of ΔKelly.
   ii There exist natural transformations $\lambda : R_Y \to R_Z$ and $\rho : R_Z \to R_Y$.

We notice that Δ, $RC(\Delta) - P$, W and $\mathcal{H}(k)$ are models of the category Kelly. Since in this category and in that of the simplicial sets the functor $(-) \times A$ has right adjoint $(\ )^A$ for each A, then in particular the cylinder functors $(-) \times \Delta[1]$ in $\Delta°S$ and $(-) \times I$ have right adjoint. It is also known that Kelly is a co complete category. At the level of realizations one thus has that

   i There exists a natural transformation $\lambda : R_{RC(\Delta)} \to R_P$ such that for each simplicial set X, $\lambda_X : R_{RC(\Delta)}(X) \to R_P(X)$ is a homotopy equivalence.

ii There exists a natural transformation $\eta : R_W \to R_P(X)$ such that for each simplicial set X, $\eta_X : R_W(X) \to R_P(X)$ is a homotopy equivalence.

iii There exists a natural transformation $\rho : R_{RC(\boldsymbol{\Delta})-P} \to R_{\boldsymbol{\Delta}} = |\ |$ that for each simplicial set X, $\rho_X : R_{RC(\boldsymbol{\Delta})-P}(X) \to |X|$ is a homotopy equivalence.

Because of parts i and ii for each simplicial set the spaces $\pi_0(X)$ (discreet), $R_{RC(\boldsymbol{\Delta})}(X)$ and $RW(X)$ are homotopically equivalent, naturally on X.

As for the singular functors since the homotopy of $\boldsymbol{\Delta}$ in Kelly is the normal homotopy then by using relations already given for singular functors one has that: *there exists a transformation $\epsilon : S_{RC(\boldsymbol{\Delta})-P} \to S_{\boldsymbol{\Delta}} = Sing$ such that for each space A of Kelly, $\epsilon_A : S_{RC(\boldsymbol{\Delta})-P}(A) \to Sing A$ is a homotopy equivalence in $\Delta^\circ S$.*

Under the context of the theory here developed parts i and iii above are not extendable to singular functors since *cosimplicial points do not carry the homotopy of $\Delta^\circ S$ into that of Kelly.* In fact suppose the opposite and consider the diagrams $(i = 0, 1)$.

When X is a simplicial point $R_P(X)$ and $R_P(X \times \Delta[1])$ have a single point. Therefore $\epsilon_X R_P(\boldsymbol{d}_X^0) = \epsilon_X R_P(\boldsymbol{d}_X^1)$, or equivalently $(*, 0) = (*, 1), * \in R_P(X)$.

In the same way W and $R_C(A)$ are homotopically equivalent but the theory cannot be applied at level of singular functors. Indeed by a similar argument to the one just given it follows that W and $RC(\boldsymbol{\Delta})$ do not carry the homotopy of $\Delta^\circ S$ in that of Kelly.

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