$p-$ADIC OSCILLATORY INTEGRALS AND NEWTON POLYHEDRA

by

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Dedicated to the Memory of Professor Jairo Charris

Resumen

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En este artículo damos una estimativa asintótica para integrales oscilantes $p-$ádicas que dependen de dos parámetros. Estas integrales son soluciones de ecuaciones seudodiferenciales $p-$ádicas del tipo Schrödinger.

Palabras clave: Cuerpos $p-$ádicos, integrales oscilatorias, funciones zeta locales de Igusa, poliedros de Newton, ecuaiones seudodiferenciales.

Abstract

In this paper we give an asymptotic estimate for $p-$adic oscilatory integrals depending of two parameters. These integrals are solutions of Schrödinger-type pseudo-differential equations.

Key words: $p-$adic fields, oscillatory integrals, Igusa local zeta function, Newton polyhedra, pseudo-differential equations.

1. Introduction

Let K be $p-$adic field, i.e. a finite extension of $\mathbb{Q}_p$, R_K the ring of integers of K, P_K the maximal ideal of R_K, and $\overline{K} = R_K/P_K$ the residue field of K. The cardinality of the residue field of K is denoted by q, thus $\overline{K} = \mathbb{F}_q$. For $z \in K$, $v(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the

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valuation of z, $|z|_K = q^{-v(z)}$, and $ac\, z = z\pi^{-v(z)}$ where π is a fixed uniformizing parameter for R_K.

Let Ψ denote a standard additive character of K, thus, for $z \in K$, $\Psi(z) = \exp\left(2\pi i Tr_{K/\mathbb{Q}_p}(z)\right)$, where Tr denotes the trace.

Let $\phi(\xi) \in R_K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$, be a non-constant polynomial, $t \in K$, with $v(t) < 0$, $x = (x_1, ..., x_n) \in K^n$, with $v(x_i) < 0$, $i = 1, ..., n,$ and

$$m = \min\left\{v(x_1), ..., v(x_n), v(t)\right\}.$$

We put $\langle x, \xi \rangle = \sum_i x_i \xi_i$, for $x, \xi \in K^N$. To these data we associate the following parametric exponential sum

$$J(x, t, \phi, K) = J(x, t)$$
$$= q^{-mn} \sum_{\xi \bmod \pi^m} \Psi\left(t\phi(\xi) + \langle x, \xi \rangle\right). \quad (1.1)$$

The exponential sum $J(x, t)$ can be expressed as an integral of the form

$$J(x, t) = \int_{R_K^n} \Psi\left(t\phi(\xi) + \langle x, \xi \rangle\right) |d\xi|, \quad (1.2)$$

where $|d\xi|$ is the Haar measure of K^n normalized so that the volume of R_K^n is 1. A more general type of oscillatory integrals is

$$I_h(x, t, \phi, K) = I(x, t)$$
$$= \int_{K^n} \Psi\left(t\phi(\xi) + \langle x, \xi \rangle\right) h(\xi) |d\xi|, \quad (1.3)$$

where h is a Bruhat-Schwartz function, i.e. a locally constant function with compact support. Integrals of the form $\int_K \Psi(a\xi + b\xi) h(\xi) |d\xi|$ are called Gaussian ones. These integrals have explicitly calculated in several cases, and they appear in certain $p-$adic quantum models [12, Chap. 1, Sect. V, and Chap. 3].

The integrals $I(x, t)$ are the non-archimedean counterpart of

$$u(x, t) = \int_{\mathbb{R}^n} \exp\left(2\pi i\left(t\phi(\xi) + \langle x, \xi \rangle\right)\right) \widehat{f}(\xi) |d\xi|, \quad (1.4)$$

where $\widehat{f}$ is the Fourier transform of f. Consider the Schrödinger–type equation

$$\frac{\partial u}{\partial t} = i\phi(D) u, \qquad u(x, 0) = f(x), \qquad (1.5)$$

here $\phi(D)$ is a pseudo-differential operator having symbol $\phi(\xi)$. Then function $u(x, t)$ is a solution for the initial value problem (1.5) (see e.g. [10, Chap. VII, VIII]).

As a consequence of the previous considerations it is natural to ask if integrals (1.2) and (1.3) satisfy some differential equation. At this point, it is important to mention that there are deep connections between differential equations and exponential sums over finite fields [5], [7].

Integrals (1.2) and (1.3) satisfy pseudo-differential equations of Schrödinger–type. Let $\mathcal{S}(K^n)$ denote the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over K^n. The dual space $\mathcal{S}'(K^n)$ is the space of distributions over K^n. A pseudo-differential operator of Schrödinger–type $A(\partial)$, with symbol $|\tau - \phi(\xi)|_K$, is an operator of the form

$$A(\partial) : \mathcal{S}(K^n) \to \mathcal{S}(K^n)$$

$$\Phi \to \mathcal{F}^{-1}_{(\xi, \tau) \to (x, t)}\left(|\tau - \phi(\xi)|_K \, \mathcal{F}_{(y, \varrho) \to (\tau, \xi)}(\Phi)\right), \quad (1.6)$$

where

$$\mathcal{F} : \mathcal{S}(K^n) \to \mathcal{S}(K^n)$$

$$\Phi \to \int_{K^n} \Psi\left(-\langle x, y \rangle\right) \Phi(x)\, dx \quad (1.7)$$

is the Fourier transform. The operator $A(\partial)$ has self-adjoint extension with dense domain in $L^2(K^n)$.

The initial value problem

$$A(\partial)z = 0, \quad z(x, 0) = h(x) \in \mathcal{S}(K^n) \quad (1.8)$$

is the non-archimedean counterpart of (1.5). By passing to the Fourier transform in (1.8), we get

$$|\tau - \phi(\xi)|_K \, \mathcal{F}_{(y, \varrho) \to (\tau, \xi)}(z) = 0, \quad (1.9)$$

from where it follows that any distribution of the form

$$z(x, t) = \left(\mathcal{F}^{-1} g\right)(x, t),$$

with $g(\xi, \tau)$ a distribution with support in

$$\left\{(\xi, \tau) \in K^{n+1} \mid |\tau - \phi(\xi)|_K = 0\right\},$$

is a solution of $A(\partial)z = 0$. In the case when

$$g(\xi, \tau) = \widehat{h}(\xi)\, \delta\left(\tau - \phi(\xi)\right),$$

where δ is the Dirac distribution, and $\widehat{h}$ is the Fourier transform of h, the distribution $z(x, t)$ takes the form (1.3). Finally, since $I(x, 0) = h(x)$, it holds that $I(x, t)$ is a solution for the initial value problem (1.8). In particular the exponential sums $J(x, t)$ satisfy (1.8), when $h(x, 0)$ is equal to the characteristic function of R_K^n.

The theory of non-archimedean pseudo-differential operators is emerging motivated for its potential use in $p-$adic physics [12], [8].

The main result of this paper (cf. Theorem 3.1) gives an asymptotic estimation of $|I(x,t)|$ for $\min\left(\|x\|_K, |t|_K\right) \to \infty$, in the case in which the singular locus of $t\phi(\xi) + \langle x, \xi \rangle$ is a subset of $\{(t, x, \xi) \mid x = \xi = 0\}$, and ϕ is a generic polynomial with an algebraically isolated singularity at the origin. The proof of the main result depends on the description of the poles of the Igusa zeta function for non-degenerate polynomials [3], [4], [13], [14], and a theorem of Igusa that establishes a connection between the poles of local zeta functions and the asymptotic expansions of certain $p-$adic oscillatory integrals [6, Theorem 8.4.2 (3)].

2. Exponential sums and Newton polyhedra

We set $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqq 0\}$. Let $f(\xi) = \sum_l a_l \xi^l \in K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$ be a polynomial in n variables satisfying $f(0) = 0$. *The Newton polyhedron* $\Gamma(f)$ of f is defined as the convex hull in $\mathbb{R}_+^n$ of the set

$$\bigcup_{m \in \{l \in \mathbb{N}^n \mid a_l \neq 0\}} \left(m + \mathbb{R}_+^n\right).$$

We denote by $\langle , \rangle$ the usual inner product of $\mathbb{R}^n$, and identify $\mathbb{R}^n$ with its dual by means of it. We set

$$\langle a_\gamma, x \rangle = m(a_\gamma),$$

for the equation of the supporting hyperplane of a facet γ (i.e. a face of codimension 1 of $\Gamma(f)$) with perpendicular vector $a_\gamma = (a_1, ..., a_n) \in \mathbb{N}^n \setminus \{0\}$, and $\sigma(a_\gamma) = \sum_i a_i$.

Definition 2.1. A polynomial $f(\xi) = \sum_i a_i \xi^i \in K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$, is called *globally non-degenerate with respect to its Newton polyhedron* $\Gamma(f)$, if it satisfies the following two properties:

(1) the origin of K^n is a singular point of $f(\xi)$;

(2) for every face $\gamma \subset \Gamma(f)$ (including $\Gamma(f)$ itself), the polynomial

$$f_\gamma(\xi) = \sum_{i \in \gamma} a_i \xi^i$$

has the property that there is no $\xi \in (K \setminus \{0\})^n$ such that

$$f_\gamma(\xi) = \frac{\partial f_\gamma}{\partial \xi_1}(\xi) = ... = \frac{\partial f_\gamma}{\partial \xi_n}(\xi) = 0.$$

For a polynomial $f(\xi) \in K[\xi]$ globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, we

set

$$\beta(f) = \max_{\tau_j}\{-\frac{\sigma(a_j)}{m(a_j)}\},$$

where τ_j runs through all facets of $\Gamma(f)$ satisfying $m(a_j) \neq 0$. We note that

$$T_0 = (-\beta(f)^{-1}, ..., -\beta(f)^{-1}) \in \mathbb{Q}^n$$

is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\{(t, ..., t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^n$.

We put

$$E(z, f, K) = E(z, f) = \int_{R_K^n} \Psi(zf(\xi)) |d\xi|,$$

with $z \in K$. Igusa showed that the asymptotic behavior of $E(z, f, K)$, when $|z|_K \to \infty$, is controlled by the largest pole of the meromorphic continuation of the local zeta function

$$Z(s, f, \chi) = \int_{R_K^n} \chi(ac\, f(\xi)) |f(\xi)|_K^s \, d\xi, \quad Re(s) > 0, \tag{2.1}$$

associated to f, and a multiplicative character χ of $R_K^\times$. More precisely, if γ_f the maximum of the real parts of the poles of $Z(s, f, \chi)$, and $\gamma > -1$, then

$$|E(z, f)| \leq C(K) |z|_K^{\gamma_f + \epsilon}, \quad \text{for} \quad |t|_K \to \infty, \tag{2.2}$$

where $C(K)$ is a constant, and $\epsilon > 0$ (see e.g. [2, Corollary 1.4.5], or [6, Theorem 8.4.2 (3)]).

Theorem 2.2. *Let K be a non-archimedean local field, and let $f(\xi) \in R_K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$, be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$. If $\beta(f) > -1$, then*

$$|E(z, f)| \leqq C(f, K) \mid z \mid_K^{\beta(f) + \epsilon}, \tag{2.3}$$

for $|z|_K \to \infty$, and any $\epsilon > 0$, here $C(f, K)$ is constant depending on f and K.

The theorem follows from (2.2) by showing the following two facts. First, the poles of $Z(s, f, \chi)$ have the form

$$s = -\frac{\sigma(a_\gamma)}{m(a_\gamma)} + \frac{2\pi i}{\log q} \frac{k}{m(a_\gamma)}, \quad k \in \mathbb{Z},$$

for some facet γ of $\Gamma(f)$ with perpendicular a_γ, and $m(a_\gamma) \neq 0$, or

$$s = -1 + \frac{2\pi i}{\log q} k, \quad k \in \mathbb{Z}.$$

Second, the maximum of the real parts of the poles of $Z(s, f, \chi_{\text{triv}})$ is $\beta(f)$, when $\beta(f) > -1$ (cf. [14,

Theorems A, B]). The description of the largest pole of $Z(s, f, \chi_{\mathrm{triv}})$ when f is non-degenerate with respect to its Newton polyhedron $\Gamma(f)$ and $\beta(f) > -1$ follows from observations made by Varchenko in [11] and was originally noted in the $p-$adic case in [9] (although it is misstated there as $\alpha(f) \neq -1$). The case $\beta(f) = -1$ is treated in [4]. The case of $\beta(f) < -1$ is more difficult and is established in [4] with some additional conditions on τ_0 by using a difficult result on exponential sums. The author proved the case $\beta(f) \geqq -1$ for polynomials with coefficients in a non-archimedean local field of arbitrary characteristic [14].

We put $\|x\|_K = \max_i \{|x_i|_K\}$, for $x \in K^n$.

Proposition 2.3. Let $f(\xi) \in R_K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$, be a non-constant polynomial without singularities on K^n. Then

$$|E(z, f)| \leq C(f, K) |z|_K^{-1+\epsilon}$$

for $|z|_K \to \infty$, and any $\epsilon > 0$, here $C(f, K)$ is a positive constant depending only on f and K.

Proof. The stationary phase formula implies that $E(z, f) = 0$ for $|z|_K \to \infty$. The proof is a slightly variation of the proof of Lemma (2.4) in [14]. $\square$

Proposition 2.4. Let $\phi(\xi) \in R_K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$, be a globally non-degenerate polynomial with respect $\Gamma(\phi)$, with $\beta(f) > -1$. If the singular locus of the polynomial $t\phi(\xi) + \langle x, \xi \rangle$ is contained in $\{(t, x, \xi) \mid x = 0\}$, then

$$|J(x, t)| \leq C(\phi, K) [\min(\|x\|_K, |t|_K)]^{\beta(\phi)+\epsilon},$$

for $\min(\|x\|_K, |t|_K) \to \infty$, and $\epsilon > 0$, here $C(\phi, K)$ a positive constant depending only on ϕ and K.

Proof. For $x \neq 0$, and $t \neq 0$, $t\phi(\xi) + \langle x, \xi \rangle$ does not have singular points on K^n. If $\frac{\|x\|_K}{|t|_K} \leq 1$, Proposition 2.3 implies that

$$|J(x, t)| \leq C_0(K) |t|_K^{-1+\epsilon},$$

for $|t|_K \to \infty$, and $\epsilon > 0$. If $\frac{|t|_K}{\|x\|_K} < 1$, Proposition 2.3 implies that

$$|J(x, t)| \leq C_1(K) \|x\|_K^{-1+\epsilon},$$

for $\|x\|_K \to \infty$, and $\epsilon > 0$. Therefore if $x \neq 0$, and $t \neq 0$,

$$|J(x, t)| \leq C_3(K) \min(\|x\|_K, |t|_K)^{-1+\epsilon}, \qquad (2.4)$$

for $\min(\|x\|_K, |t|_K) \to \infty$, and $\epsilon > 0$.

For $x = 0$, and $t \neq 0$, Theorem 2.2 implies that

$$|J(x, t)| \leq C_4(\phi, K) |t|_K^{\beta(\phi)+\epsilon}, \qquad (2.5)$$

for $|t|_K \to \infty$, and $\epsilon > 0$. For $x \neq 0$, and $t = 0$,

$$J(x, t) = 0, \qquad (2.6)$$

for $|t|_K \geq 1$. Since $\beta(f) > -1$, estimates (2.4), (2.5), and (2.6) imply that

$$|J(x, t)| \leq C(\phi, K) [\min(\|x\|_K, |t|_K)]^{\beta(\phi)+\epsilon},$$

for $\min(\|x\|_K, |t|_K) \to \infty$, and $\epsilon > 0$. $\square$

The proof of the following Proposition is similar to the previous one.

Proposition 2.5. Let $\phi(\xi) \in R_K[\xi]$, $\xi = (\xi_1, ..., \xi_n)$, be polynomial such that $t\phi(\xi) + \langle x, \xi \rangle$ has no singular points on K^n, for any $t \in K$, and $x \in K^n$, then

$$|J(x, t)| \leq C(K) [\min(\|x\|_K, |t|_K)]^{-1+\epsilon},$$

for $\min(\|x\|_K, |t|_K) \to \infty$, and $\epsilon > 0$, here $C(K)$ a positive constant.

3. Main Result

We shall say that the origin of K^n is an *algebraically isolated singularity* of $\phi(\xi) \in K[\xi_1, ..., \xi_n]$, if the origin is the only solution of the system

$$\phi(\xi) = \frac{\partial \phi}{\partial \xi_1}(\xi) = ... = \frac{\partial \phi}{\partial \xi_n}(\xi) = 0. \qquad (3.1)$$

Theorem 3.1. Let $\phi(\xi) \in R_K[\xi_1, ..., \xi_n]$ be a non-constant polynomial with an algebraically isolated singularity at the origin, such that $\phi(\xi)$ is globally non-generate with respect $\Gamma(\phi)$, and $\beta(\phi) > -1$. If the singular locus of the polynomial $t\phi(\xi) + \langle x, \xi \rangle$ is contained in $\{(t, x, \xi) \mid x = 0\}$, then

$$|I(x, t)| \leq C(\phi, K) [\min(\|x\|_K, |t|_K)]^{\beta(\phi)+\epsilon}, \qquad (3.2)$$

for $\min(\|x\|_K, |t|_K) \to \infty$, and any $\epsilon > 0$, here $C(\phi, K)$ a positive constant depending only on ϕ and K.

Proof. Let $\cup_i (z_i + \pi^{e_0} R_K^n)$ be a finite covering of the support of h such that $h\mid_{z_i + \pi^{e_0} R_K^n} = h(z_i)$. Then

$$I(x, t) = \sum_i c_i \int_{R_K^n} \Psi(t\phi(z_i + \pi^{e_0}\xi) + \langle \pi^{e_0}\xi, x \rangle) |d\xi|,$$

$$(3.3)$$

where $c_i = q^{-e_0 n} h(z_i) \Psi(\langle z_i, x \rangle)$. We put $\phi(z_i + \pi^{e_0}\xi) = \phi(z_i) + \pi^{e_0}\phi_i^*(\xi)$, with $\phi_i^*(\xi) \in K[\xi]$. Without loss of generality we may assume that $\pi^{e_0}\phi_i^*(\xi) \in R_K[\xi]$. With this notation, (3.3) can be rewritten as

$$I(x,t) = \sum_i c_i \Psi\left(t\phi\left(z_i\right)\right) J(x, t, \pi^{e_0}\phi_i^*\left(\xi\right)). \qquad (3.4)$$

If z_i is the origin then

$$|J(x, t, \pi^{e_0}\phi_i^*\left(\xi\right))| \le$$
$$C_1(\phi, K)\left[\min\left(\|x\|_K, |t|_K\right)\right]^{\beta(\phi)+\epsilon}, \tag{3.5}$$

for $\min\left(\|x\|_K, |t|_K\right) \to \infty$, and any $\epsilon > 0$, with $C_1(\phi, K)$ a positive constant depending only on h and K (cf. Proposition 2.4).

If z_i is not the origin, then $\pi^{e_0}\phi_i^*\left(\xi\right)$ does not have singularities on K^n. Then Proposition 2.5 implies that

$$|J(x, t, \pi^{e_0}\phi_i^*\left(\xi\right))|_{\mathbb{C}} \le$$
$$C_2(K)\left[\min\left(\|x\|_K, |t|_K\right)\right]^{-1+\epsilon}, \tag{3.6}$$

for $\min\left(\|x\|_K, |t|_K\right) \to \infty$, and any $\epsilon > 0$, with $C_2(K)$ a positive constant. The result follows from (3.5) and (3.6) by using the fact that $\beta\left(\phi\right) > -1$.

3.1. Remarks

1. The main result is valid for non-archimedean local fields of positive characteristic (cf. [14, Theorems A, B, and Corollary 6.1]).
2. Recently R. Cluckers showed that

$$\left| \int_{R_K^n} \Psi\left(\langle y, f\left(\xi\right)\rangle\right) |d\xi| \right|_{\mathbb{C}} \le C \|y\|_K^\alpha, \; \alpha < 0,$$

for $\|y\|_K \to \infty$, when $f\left(\xi\right) = (f_1\left(\xi\right), ..., f_n\left(\xi\right))$ is a dominant polynomial map. This result does not provide any information about α [1, Chap. VI].

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