LEVERRIER-FADEEV ALGORITHM AND CLASSICAL ORTHOGONAL POLYNOMIALS

by

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To Professor Jairo Charris Castañeda, as a tribute of our mathematical friendship

Resumen

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Usando propiedades estructurales de los polinomios ortogonales clásicos (Hermite, Laguerre, Jacobi y Bessel), se implementa el algoritmo de Leverrier-Fadeev para obtener el polinomio característico de una matriz cuadrada de elementos complejos.

Palabras clave: Polinomio característico, funciones de transferencia, polinomios ortogonales, funcionales lineales clásicos.

Abstract

Using structural properties of classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel), an implementation of Leverrier-Fadeev algorithm to obtain the characteristic polynomial of a square matrix with complex entries is presented.

Key words: Characteristic Polynomial, Transfer Functions, Orthogonal Polynomials, Classical Linear Functionals.

1. Introduction

For a given matrix $A \in \mathbb{C}^{n\times n}$ an algorithm attributed to Leverrier, Fadeev, and others, allows the simultaneous determination of the characteristic polynomial of A and the adjoint matrix of $sI - A$, where I denotes the identity matrix in $\mathbb{C}^{n\times n}$. Indeed, if

$$p(s) = \det(sI - A) = s^n + \sum_{k=0}^{n-1} a_{n-k}s^k$$

denotes the characteristic polynomial of A and

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$$\tilde{A}(s) = \text{Adj }(sI - A) = s^{n-1}I + \sum_{k=0}^{n-2} s^k B_{n-k-1}$$

denotes the adjoint matrix of $sI - A$, and taking into account

$$\tilde{A}(s) = p(s)(sI - A)^{-1},$$

then the coefficients (a_k) and the matrices (B_k) can be generated from

$$\begin{aligned} a_1 &= -\text{tr } A, \qquad & B_1 &= A + a_1 I, \\ a_k &= -\tfrac{1}{k}\text{tr }(AB_{k-1}), \quad & B_k &= a_k I + AB_{k-1}, \end{aligned} \tag{1.1}$$

for $k = 2, \dots, n-1$. Here tr A denotes the trace of the matrix A.

Notice that (1.1) can be read as follows (See [3])

$$\begin{cases} (sI - A)\tilde{A}(s) = p(s)I, \\ \dfrac{dp(s)}{ds} = \text{tr } \tilde{A}(s). \end{cases} \tag{1.2}$$

Despite the little value from a numerical point of view, this algorithm is useful for theoretical purposes as well as for the applications in linear control theory. More precisely, $\dfrac{1}{p(s)}\tilde{A}(s)$ is the transfer function of a continuous time linear system with n inputs and n outputs.

The algorithm takes into account the representation of the characteristic polynomial and the adjoint matrix in terms of the canonical basis $\left\{s^k\right\}_{k=0}^{n}$ in the linear space of polynomials with complex coefficients and degree at most n.

From a computational point of view the accuracy of the algorithm using an orthogonal polynomial system is improved. For some particular cases of orthogonal polynomials S. Barnett [1] gave an implementation of the algorithm. The key idea is the relation (1.2) as well as the expression of the derivative of the polynomial P_k, $k = 1, \dots, n$, in terms of the family $\{P_k\}_{k=0}^{n}$. The aim of our contribution is to present a general approach for families of classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) taking into account a characterization of such families obtained in [4]. Indeed it allows to give an expression of P_k as a linear combination of P'_{k+1}, P'_k, and P'_{k-1}. Thus we can show a very simple implementation of the Leverrier algorithm, where parameters associated with the three-term recurrence relation play the main role.

The structure of the paper is the following. In the section 2 we summarize the basic properties of classical orthogonal polynomials. In the section 3 we present the adapted version of Leverrier algorithm for bases of classical orthogonal polynomials, and we analyze it for each family of classical orthogonal polynomials. In the section 4, some examples are tested.

2. Classical Orthogonal Polynomials

Let u be a linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients. If $\langle\ ,\ \rangle$ denotes the duality bracket then $c_n = \langle u, x^n\rangle$ is said to be the moment of order n associated with the linear functional u.

The linear functional u is said to be quasi-definite [2] if the principal submatrices of the Hankel matrix $H = (c_{i+j})_{i,j=0}^{\infty}$ are non-singular. In such a case, there exists a unique sequence of monic polynomials $\{P_n\}_{n=0}^{\infty}$ such that

(i) $\langle u, x^k P_n\rangle = 0, \quad k = 0, 1, \dots, n-1.$
(ii) $\langle u, x^n P_n\rangle \neq 0.$
(iii) $\deg P_n = n.$

The sequence $\{P_n\}_{n=0}^{\infty}$ is said to be a sequence of monic orthogonal polynomials (SMOP) with respect to u. It is very well known that $\{P_n\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \tag{2.1}$$

$n = 1, 2, \dots$ with $\gamma_n \neq 0$.

The converse is also true and this result is due to several authors despite the fact is known as Favard's Theorem [2].

If $q(x)$ denotes a polynomial, then a new linear functional $\tilde{u} = q(x)u$ can be introduced as follows

$$\langle \tilde{u}, p(x)\rangle = \langle u, p(x)q(x)\rangle \tag{2.2}$$

for every $p \in \mathbb{P}$.

On the other hand, as for a distribution, the derivative of the linear functional u, Du, is given by $\langle Du, p(x)\rangle = -\langle u, p'(x)\rangle, \quad p \in \mathbb{P}$.

Definition 2.1. A linear functional u is said to be classical if there exist polynomials ϕ, ψ, with $\deg\phi \leqslant 2$ and $\deg\psi = 1$ such that

$$D(\phi u) = \psi u. \tag{2.3}$$

Up to a linear change of variables, four cases appear

(i) $\phi(x) = 1$. This leads to Hermite linear functional with $\psi(x) = -2x$.
(ii) $\phi(x) = x$. This leads to Laguerre linear functional with $\psi(x) = -x + \alpha + 1$.
(iii) $\phi(x) = x^2 - 1$. This yields the Jacobi linear functional with $\psi(x) = -(\alpha + \beta + 2)x + \beta - \alpha$.
(iv) $\phi(x) = x^2$. This yields the Bessel linear functional with $\psi(x) = (\alpha + 2)x + 2$.

Theorem 2.2.(see [4]) *If $\{P_n\}_{n=0}^{\infty}$ is the SMOP associated with u, then the following statements are equivalent*

(i) *u is a classical linear functional.*
(ii) *$\{Q_n\}_{n=0}^{\infty}$, with $Q_n = \dfrac{P'_{n+1}}{n+1}$, is a SMOP.*
(iii) $P_n = Q_n + r_n Q_{n-1} + s_n Q_{n-2}$.
(iv) $\phi(x)Q_n = a_n P_{n+2} + b_n P_{n+1} + c_n P_n$, *with* $c_n \neq 0$.

Tabla 1. Coefficients in the three-term recurrence relation (2.1)

	β_n	γ_n
Hermite	0	$\frac{n}{2}$
Laguerre	$2n + \alpha + 1$	$n(n + \alpha)$
Jacobi	$\frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}$	$\frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}$
Bessel	$-\frac{2\alpha}{(2n + \alpha)(2n + \alpha + 2)}$	$-\frac{4n(n + \alpha)}{(2n + \alpha - 1)(2n + \alpha)^2(2n + \alpha + 1)}$

Tabla 2. Coefficients in the relation of the Theorem 2.2 (iii)

	r_n	s_n
Hermite	0	0
Laguerre	n	0
Jacobi	$\frac{2n(\alpha - \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}$	$-\frac{4n(n-1)(n + \alpha)(n + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}$
Bessel	$\frac{4n}{(2n + \alpha)(2n + \alpha + 2)}$	$\frac{4n(n-1)}{(2n + \alpha - 1)(2n + \alpha)^2(2n + \alpha + 1)}$

3. Leverrier-Fadeev Algorithm

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials. If we expand the characteristic polynomial $p(s)$ and the adjoint matrix $\tilde{A}(s)$ of a matrix $A \in \mathbb{C}^{n \times n}$ in terms of the above basis in the linear space of polynomials with complex coefficients, then we get:

$$p(s) = P_n(s) + \sum_{k=0}^{n-1} \hat{a}_{n-k} P_k(s), \tag{3.1}$$

$$\tilde{A}(s) = P_{n-1}(s)I + \sum_{k=0}^{n-2} P_k(s)\hat{B}_{n-k-1}. \tag{3.2}$$

From the first identity in (1.2)

$$(sI - A)\left(P_{n-1}(s)I + \sum_{k=0}^{n-2} P_k(s)\hat{B}_{n-k-1}\right) = P_n(s)I + \sum_{k=0}^{n-1} \hat{a}_{n-k}P_k(s)I. \tag{3.3}$$

Taking into account the three-term recurrence relation (2.1) for the family $\{P_n\}_{n=0}^{\infty}$, (3.3) becomes

$$P_n(s)I + \sum_{k=0}^{n-1} \hat{a}_{n-k}P_k(s)I = [P_n(s) + \beta_{n-1}P_{n-1} + \gamma_{n-1}P_{n-2}]\, I - P_{n-1}(s)A + \sum_{k=0}^{n-2} (P_{k+1}(s) + \beta_k P_k(s) + \gamma_k P_{k-1}(s))\, \hat{B}_{n-k-1} - \sum_{k=0}^{n-2} P_k(s)A\hat{B}_{n-k-1}\ .$$

Equating coefficients of P_k in the previous expression we get

$$\begin{array}{rcll} A\hat{B}_0 &=& -\hat{a}_1 I + \beta_{n-1}\hat{B}_0 + \hat{B}_1, & \\ A\hat{B}_1 &=& -\hat{a}_2 I + \gamma_{n-1}\hat{B}_0 + \beta_{n-2}\hat{B}_1 + \hat{B}_2, & \\ \vdots & & & \\ A\hat{B}_{n-k-1} &=& -\hat{a}_{n-k} I + \gamma_{k+1}\hat{B}_{n-k-2} + \beta_k\hat{B}_{n-k-1} + \hat{B}_{n-k}, & k = 1, 2, \dots, n-3, \\ A\hat{B}_{n-1} &=& -\hat{a}_n I + \gamma_1\hat{B}_{n-2} + \beta_0\hat{B}_{n-1}, & \end{array} \tag{3.4}$$

with $\hat{B}_0 = I$. In a matrix form

$$A\begin{bmatrix} \hat{B}_{n-1} \\ \vdots \\ \hat{B}_0 \end{bmatrix} = M\begin{bmatrix} \hat{B}_{n-1} \\ \vdots \\ \hat{B}_0 \end{bmatrix}$$

where $M = J_n - [0|\hat{a}]$. J_n is the Jacobi matrix of dimension n associated with the SMOP $\{P_n\}_{n=0}^{\infty}$ i. e.

$$J_n = \begin{bmatrix} \beta_0 & \gamma_1 & 0 & \cdots & 0 \\ 1 & \beta_1 & \gamma_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & & & \gamma_{n-1} \\ 0 & \cdots & 0 & 1 & \beta_{n-1} \end{bmatrix}$$

and

$$\hat{a} = \begin{bmatrix} \hat{a}_n \\ \hat{a}_{n-1} \\ \vdots \\ \hat{a}_1 \end{bmatrix}.$$

In the literature, the matrix M is called the comrade matrix of A with respect to the orthogonal system $\{P_n\}_{n=0}^{\infty}$. His characteristic polynomial is $p(s)$. In particular, we get

$$\operatorname{tr} A = -\sum_{j=0}^{n-1} \beta_j - \hat{a}_1.$$

On the other hand, from the second relation in (1.2) for $n = 2, 3, \dots$ we have

$$P_n'(s) + \sum_{k=0}^{n-1} \hat{a}_{n-k}P_k'(s) = nP_{n-1}(s) + \sum_{k=0}^{n-2} P_k(s)\operatorname{tr}\, \hat{B}_{n-k-1}. \tag{3.5}$$

If $\{P_n\}_{n=0}^{\infty}$ is a classical family then, from theorem (iii), we get

$$P_k(s) = \frac{P_{k+1}'(s)}{k+1} + r_k\frac{P_k'(s)}{k} + s_k\frac{P_{k-1}'(s)}{k-1},\ k = 2, 3, \dots$$

Thus, substitution in (3.5) yields

$$\begin{aligned} P'_n(s) + \sum_{k=0}^{n-1} \hat{a}_{n-k} P'_k(s) &= P'_n(s) + r_{n-1}\frac{n}{n-1}P'_{n-1}(s) + s_{n-1}\frac{n}{n-2}P'_{n-2}(s) + \\ &+ \sum_{k=2}^{n-2}\left(\frac{P'_{k+1}(s)}{k+1} + r_k\frac{P'_k(s)}{k} + s_k\frac{P'_{k-1}(s)}{k-1}\right)\text{tr }\hat{B}_{n-k-1} + \\ &+ \text{tr }\hat{B}_{n-1}P'_1(s) + \text{tr }\hat{B}_{n-2}\left(\frac{P'_2(s)}{2} + r_1 P'_1(s)\right). \end{aligned}$$

Finally, equating the coefficients of P'_k in both hand sides we get

$$\begin{aligned} (n-1)\hat{a}_1 &= n r_{n-1} + \text{tr }\hat{B}_1, \\ (n-2)\hat{a}_2 &= n s_{n-1} + r_{n-2}\text{tr }\hat{B}_1 + \text{tr }\hat{B}_2, \\ &\vdots \\ k\hat{a}_{n-k} &= s_{k+1}\text{tr }\hat{B}_{n-k-2} + r_k\text{tr }\hat{B}_{n-k-1} + \text{tr }\hat{B}_{n-k}, \qquad k = 1, 2, \ldots, n-3. \end{aligned} \tag{3.6}$$

Thus, in order to obtain $(\hat{a}_k)$ and $(\hat{B}_k)$ we will proceed as follows.

First Step

$$\hat{a}_1 = n\left(\beta_{n-1} - r_{n-1}\right) - \text{tr }A. \tag{3.7}$$

Indeed, taking traces in the first equation of (3.4), and (3.6)

$$\begin{cases} \text{tr }A = -n\hat{a}_1 + n\beta_{n-1} + \text{tr }\hat{B}_1, \\ (n-1)\hat{a}_1 = n r_{n-1} + \text{tr }\hat{B}_1, \end{cases}$$

and (3.7) follows.

Second Step

$$\hat{B}_1 = A\hat{B}_0 + \hat{a}_1 I - \beta_{n-1}\hat{B}_0. \tag{3.8}$$

Third Step

$$2\hat{a}_2 = (\gamma_{n-1} - s_{n-1})\text{tr }\hat{B}_0 + (\beta_{n-2} - r_{n-2})\text{tr }\hat{B}_1 - \text{tr }\left(A\hat{B}_1\right). \tag{3.9}$$

Indeed, from the second equation in (3.4) and (3.6)

$$\begin{cases} \text{tr }\left(A\hat{B}_1\right) = n\left(\gamma_{n-1} - \hat{a}_2\right) + \beta_{n-2}\text{tr }\hat{B}_1 + \text{tr }\hat{B}_2, \\ \text{tr }\hat{B}_2 = (n-2)\hat{a}_2 - n s_{n-1} - r_{n-2}\text{tr }\hat{B}_1, \end{cases}$$

and (3.9) follows.

Fourth Step

$$\hat{B}_2 = A\hat{B}_1 + \hat{a}_2 I - \gamma_{n-1}\hat{B}_0 - \beta_{n-2}\hat{B}_1. \tag{3.10}$$

Thus, for $k = 1, 2, \ldots, n-3$,

$$(n-k)\hat{a}_{n-k} = (\beta_k - r_k)\text{tr }\hat{B}_{n-k-1} + (\gamma_{k+1} - s_{k+1})\text{tr }\hat{B}_{n-k-2} - \text{tr }\left(A\hat{B}_{n-k-1}\right), \tag{3.11}$$

as well as

$$\hat{B}_{n-k} = A\hat{B}_{n-k-1} + \hat{a}_{n-k}I - \gamma_{k+1}\hat{B}_{n-k-2} - \beta_k\hat{B}_{n-k-1}.$$

These results follow from the expressions in (3.4) and (3.6) for $k = 1, \ldots, n-3$.

Finally, taking traces in the last equation of (3.4) we get

$$n\hat{a}_n = \beta_0\text{tr }\hat{B}_{n-1} + \gamma_1\text{tr }\hat{B}_{n-2} - \text{tr }\left(A\hat{B}_{n-1}\right).$$

As a conclusion we get

Theorem 3.1.

(i) *For* $k = 0, 1, \ldots, n-1$,

$$(n-k)\hat{a}_{n-k} = (\beta_k - r_k)\text{tr }\hat{B}_{n-k-1} + (\gamma_{k+1} - s_{k+1})\text{tr }\hat{B}_{n-k-2} - \text{tr }\left(A\hat{B}_{n-k-1}\right), \tag{3.12}$$

with the convention $\hat{B}_{-1} = 0,\ r_0 = 0,\ s_1 = 0.$

(ii) *For* $k = 1, 2, \ldots, n-1$

$$\hat{B}_{n-k} = A\hat{B}_{n-k-1} + \hat{a}_{n-k}I - \gamma_{k+1}\hat{B}_{n-k-2} - \beta_k\hat{B}_{n-k-1}. \tag{3.13}$$

The implementation of the algorithm is as follows

DATA: $\{\beta_k\}_{k=0}^{n-1},\ \{\gamma_k\}_{k=1}^{n},\ \{r_k\}_{k=0}^{n-1},\ \{s_k\}_{k=1}^{n}$.

Initial Condition: $\hat{B}_{-1}=0,\ \hat{B}_0=I$.

1. From $\hat{B}_{n-k-2}$ and $\hat{B}_{n-k-1}$ taking into account (3.12) we get $\hat{a}_{n-k}$.
2. From (3.13) we get $\hat{B}_{n-k}$.

END

and, for each family of monic orthogonal polynomials, is given in below

3.1. Hermite Case. According to Theorem 3.1 we get

(i) $(n-k)\hat{a}_{n-k}=\frac{k+1}{2}\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right)$.

$$\text{(ii)}\hat{B}_{n-k}=A\hat{B}_{n-k-1}+\hat{a}_{n-k}I-\frac{k+1}{2}\hat{B}_{n-k-2}. \quad (3.14)$$

In particular, taking traces in (ii) and using (i) we get

$$\mathrm{tr}\ \hat{B}_{n-k}=k\hat{a}_{n-k}.$$

This is formula (3.12) in [1].

Furthermore, substituting in (i) we get

$$(n-k)\hat{a}_{n-k}=\frac{(k+1)(k+2)}{2}\hat{a}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right), \quad (3.15)$$

i.e.

$$\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right)=\frac{(k+1)(k+2)}{2}\hat{a}_{n-k-2}-(n-k)\hat{a}_{n-k}.$$

3.2. Laguerre Case. According to Theorem 3.1 we get

$$\text{(i)}\ (n-k)\hat{a}_{n-k}=[(2k+\alpha+1)-k]\ \mathrm{tr}\ \hat{B}_{n-k-1}+(k+1)(k+\alpha+1)\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right)$$

i.e.

$$(n-k)\hat{a}_{n-k}=(k+\alpha+1)\mathrm{tr}\ \hat{B}_{n-k-1}+(k+1)(k+\alpha+1)\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right)$$

$$\text{(ii)}\ \hat{B}_{n-k}=A\hat{B}_{n-k-1}+\hat{a}_{n-k}I-(k+1)(k+\alpha+1)\hat{B}_{n-k-2}-(2k+\alpha+1)\hat{B}_{n-k-1} \quad (3.16)$$

Taking traces in (ii) and using (i) we get

$$\mathrm{tr}\ \hat{B}_{n-k}=k\hat{a}_{n-k}-k\mathrm{tr}\ \hat{B}_{n-k-1}.$$

Thus we deduce

$$(n-k)\hat{a}_{n-k}=(k+\alpha+1)\mathrm{tr}\ \hat{B}_{n-k-1}+(k+\alpha+1)\left[(k+1)\hat{a}_{n-k-1}-\mathrm{tr}\ \hat{B}_{n-k-1}\right]-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right),$$

i.e.

$$(n-k)\hat{a}_{n-k}=(k+\alpha+1)(k+1)\hat{a}_{n-k-1}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right). \quad (3.17)$$

Up to a normalization this is the formula (3.23b) in [1], when $\alpha=0$.

3.3. Jacobi Case. According to Theorem 3.1 we get

$$(n-k)\hat{a}_{n-k}=\left(\frac{\beta^2-\alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}-\frac{2k(\alpha-\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}\right)\mathrm{tr}\ \hat{B}_{n-k-1}+\left(\frac{4(k+1)(k+1+\alpha)(k+1+\beta)(k+1+\alpha+\beta)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)^2(2k+\alpha+\beta+3)}-\frac{4k(k+1)(k+1+\alpha)(k+1+\beta)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)^2(2k+\alpha+\beta+3)}\right)\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right)=\frac{\beta-\alpha}{2k+\alpha+\beta+2}\mathrm{tr}\ \hat{B}_{n-k-1}+\frac{4(k+1)(k+1+\alpha)(k+1+\beta)}{(2k+\alpha+\beta+2)^2(2k+\alpha+\beta+3)}\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right).$$

On the other hand, if $\alpha=\beta$ then we are in the Gegenbauer case. The linear functional is symmetric and thus get

$$\begin{aligned}(n-k)\hat{a}_{n-k}&=\frac{4(k+1)(k+1+\alpha)^2}{(2k+2\alpha+2)^2(2k+2\alpha+3)}\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right)\\&=\frac{k+1}{2k+2\alpha+3}\mathrm{tr}\ \hat{B}_{n-k-2}-\mathrm{tr}\ \left(A\hat{B}_{n-k-1}\right),\end{aligned} \quad (3.18)$$

or, equivalently

$$\operatorname{tr}\hat{B}_{n-k} = k\hat{a}_{n-k} + \frac{4k(k+1)(k+1+\alpha)^2}{(2k+2\alpha+1)(2k+2\alpha+2)^2(2k+2\alpha+3)}\operatorname{tr}\hat{B}_{n-k-2},$$

i.e.

$$\operatorname{tr}\hat{B}_{n-k} = k\hat{a}_{n-k} + \frac{k(k+1)}{(2k+2\alpha+1)(2k+2\alpha+3)}\hat{B}_{n-k-2}.$$

Notice that the symmetry of the linear functional yields an important simplification in our algorithm.

Furthermore

$$\begin{aligned}\hat{B}_{n-k} &= A\hat{B}_{n-k-1} + \hat{a}_{n-k}I - \frac{4(k+1)(k+\alpha+1)^2(k+2\alpha+1)}{(2k+2\alpha+1)(2k+2\alpha+2)^2(2k+2\alpha+3)}\hat{B}_{n-k-2} \\ &= A\hat{B}_{n-k-1} + \hat{a}_{n-k}I - \frac{(k+1)(k+2\alpha+1)}{(2k+2\alpha+1)(2k+2\alpha+3)}\hat{B}_{n-k-2}.\end{aligned} \tag{3.19}$$

This is, up to the corresponding normalization, the formula (3.20) in [1] for $\alpha = 0$.

3.4. Bessel Case. According to Theorem 3.1 we get

$$\begin{aligned}(n-k)\hat{a}_{n-k} &= \frac{-2\alpha-4k}{(2k+\alpha)(2k+\alpha+2)}\operatorname{tr}\hat{B}_{n-k-1} - \frac{4(k+1)(2k+\alpha+1)}{(2k+\alpha+1)(2k+\alpha+2)^2(2k+\alpha+3)}\operatorname{tr}\hat{B}_{n-k-2} - \operatorname{tr}\left(A\hat{B}_{n-k-1}\right) \\ &= \frac{-2}{2k+\alpha+2}\operatorname{tr}\hat{B}_{n-k-1} - \frac{4(k+1)}{(2k+\alpha+2)^2(2k+\alpha+3)}\operatorname{tr}\hat{B}_{n-k-2} - \operatorname{tr}\left(A\hat{B}_{n-k-1}\right),\end{aligned}$$

i.e.

$$(n-k)\hat{a}_{n-k} + \frac{1}{k+1+\frac{\alpha}{2}}\operatorname{tr}\hat{B}_{n-k-1} + \operatorname{tr}\left(A\hat{B}_{n-k-1}\right) + \frac{k+1}{\left(k+1+\frac{\alpha}{2}\right)^2(2k+\alpha+3)}\operatorname{tr}\hat{B}_{n-k-2} = 0, \tag{3.20}$$

together with

$$\hat{B}_{n-k} = A\hat{B}_{n-k-1} + \hat{a}_{n-k}I + \frac{4(k+1)(k+\alpha+1)}{(2k+\alpha+1)(2k+\alpha+2)^2(2k+\alpha+3)}\hat{B}_{n-k-2} + \frac{2\alpha}{(2k+\alpha)(2k+\alpha+2)}\hat{B}_{n-k-1}. \tag{3.21}$$

4. Example

Consider

$$A = \begin{bmatrix} 1 & -4 & -1 & -1 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$$

which has characteristic polynomial

$$a(s) = s^4 - 5s^3 + 9s^2 - 7s + 2.$$

We apply the algorithm for each basis.

4.1. Hermite Basis. From (3.15), $a_1 = -\operatorname{tr} A = -5$, and from (3.14)

$$B_1 = a_1 I + A = \begin{bmatrix} -4 & -4 & -1 & -4 \\ 2 & -5 & 5 & -4 \\ -1 & 1 & -7 & 3 \\ -1 & 4 & -1 & 1 \end{bmatrix}.$$

Using (3.15), we get

$$a_2 = 3 - \frac{1}{2}\operatorname{tr}(AB_1) = 12,$$

and from (3.14)

$$B_2 = a_2 I - \frac{3}{2}B_0 + AB_1 = \begin{bmatrix} \frac{7}{2} & -1 & -10 & 5 \\ -9 & -\frac{17}{2} & -33 & 3 \\ 5 & 9 & \frac{55}{2} & -3 \\ 7 & 7 & 22 & \frac{3}{2} \end{bmatrix}.$$

Using again (3.15)

$$a_3 = a_1 - \frac{1}{3}\operatorname{tr}(AB_2) = -\frac{29}{2},$$

and from (3.14)

$$B_3 = a_3 I - B_1 + AB_2 = \frac{1}{2}\begin{bmatrix} -8 & 0 & 15 & -12 \\ 4 & 11 & 49 & -14 \\ -1 & -11 & -39 & 11 \\ -3 & -8 & -33 & 7 \end{bmatrix}.$$

Finally

$$a_4 = \frac{1}{4}a_2 - \frac{1}{4}\text{tr } (AB_3) = \frac{29}{4}.$$

Hence, the characteristic polynomial of A is given by (3.1) as

$$a(s) = H_4(s) - 5H_3(s) + 12H_2(s) - \frac{29}{2}H_1(s) + \frac{29}{4}H_0(s).$$

4,2, Laguerre Basis. We consider the family $\{L_n^0\}_{n=0}^{\infty}$. From (3.17), $a_1 = 16 - \text{tr } A = 11,$ and from (3.16)

$$B_1 = a_1 I - 7B_0 + A = \begin{bmatrix} 5 & -4 & -1 & -4 \\ 2 & 4 & 5 & -4 \\ -1 & 1 & 2 & 3 \\ -1 & 4 & -1 & 10 \end{bmatrix}.$$

Using (3.17), we get

$$a_2 = \frac{9}{2}a_1 - \frac{1}{2}\text{tr } (AB_1) = 36,$$

and from (3.16)

$$B_2 = a_2 I - 9B_0 - 5B_1 + AB_1 = \begin{bmatrix} 4 & -17 & -14 & -11 \\ -1 & -12 & -13 & -13 \\ 1 & 13 & 16 & 9 \\ 3 & 23 & 18 & 22 \end{bmatrix}.$$

Using again (3.17)

$$a_3 = \frac{4}{3}a_2 - \frac{1}{3}\text{tr } (AB_2) = 35,$$

and from (3.16)

$$B_3 = a_3 I - 4B_1 - 3B_2 + AB_2 = \begin{bmatrix} -2 & -7 & -4 & -7 \\ -4 & -6 & -1 & -10 \\ 3 & 5 & 2 & 7 \\ 4 & 9 & 4 & 11 \end{bmatrix}.$$

Finally

$$a_4 = \frac{1}{4}a_3 - \frac{1}{4}\text{tr } (AB_3) = 7.$$

Hence, the characteristic polynomial of A is given by (3.1) as

$$a(s) = L_4^0(s) + 11L_3^0(s) + 36L_2^0(s) + 35L_1^0(s) + 7L_0^0(s).$$

4.3. Jacobi Basis. We consider the family $P_n = P_n^{(0,0)}$ (Legendre Polynomials). From (3.18), $a_1 = -\text{tr } A = -5,$ and from (3.19)

$$B_1 = a_1 I + A = \begin{bmatrix} -4 & -4 & -1 & -4 \\ 2 & -5 & 5 & -4 \\ -1 & 1 & -7 & 3 \\ -1 & 4 & -1 & 1 \end{bmatrix}.$$

Using (3.18), we get

$$a_2 = \frac{6}{7} - \frac{1}{2}\text{tr } (AB_1) = \frac{69}{7},$$

and from (3.19)

$$B_2 = a_2 I - \frac{9}{35}B_0 + AB_1 = \begin{bmatrix} \frac{13}{5} & -1 & -10 & 5 \\ -9 & -\frac{47}{5} & -33 & 3 \\ 5 & 9 & \frac{133}{5} & -3 \\ 7 & 7 & 22 & \frac{3}{5} \end{bmatrix}.$$

Using again (3.18)

$$a_3 = \frac{2}{15}\text{tr } B_1 - \frac{1}{3}\text{tr } (AB_2) = -10,$$

and from (3.19)

$$B_3 = a_3 I - \frac{4}{15}B_1 + AB_2 = \frac{1}{3}\begin{bmatrix} -10 & 2 & 23 & -16 \\ 5 & 19 & 71 & -19 \\ -1 & -17 & -55 & 15 \\ -4 & -14 & -49 & 10 \end{bmatrix}.$$

Finally

$$a_4 = \frac{1}{12}\text{tr } B_2 - \frac{1}{4}\text{tr } (AB_3) = \frac{26}{5}.$$

Hence, the characteristic polynomial of A is given by (3.1) as

$$a(s) = P_4(s) - 5P_3(s) + \frac{69}{7}P_2(s) - 10P_1(s) + \frac{26}{5}P_0(s).$$

Now, We consider the family $U_n = P_n^{(\frac{1}{2},\frac{1}{2})}$ (Chebyshev Polynomials of the second kind). From (3.18), $a_1 = -\text{tr } A = -5,$ and from (3.19)

$$B_1 = a_1 I + A = \begin{bmatrix} -4 & -4 & -1 & -4 \\ 2 & -5 & 5 & -4 \\ -1 & 1 & -7 & 3 \\ -1 & 4 & -1 & 1 \end{bmatrix}.$$

Using (3.18), we get

$$a_2 = \frac{3}{4} - \frac{1}{2}\text{tr } (AB_1) = \frac{39}{4},$$

and from (3.19)

$$B_2 = a_2 I - \frac{1}{4} B_0 + AB_1 = \begin{bmatrix} \frac{5}{2} & -1 & -10 & 5 \\ -9 & -\frac{19}{2} & -33 & 3 \\ 5 & 9 & \frac{53}{2} & -3 \\ 7 & 7 & 22 & \frac{1}{2} \end{bmatrix}.$$

Using again (3.18)

$$a_3 = \frac{1}{9} \text{tr } B_1 - \frac{1}{3} \text{tr } (AB_2) = -\frac{19}{2},$$

and from (3.19)

$$B_3 = a_3 I - \frac{1}{4} B_1 + AB_2 = \frac{1}{4} \begin{bmatrix} -12 & 4 & 31 & -20 \\ 6 & 27 & 93 & -24 \\ -1 & -23 & -71 & 19 \\ -5 & -20 & -65 & 13 \end{bmatrix}.$$

Finally

$$a_4 = \frac{1}{16} \text{tr } B_2 - \frac{1}{4} \text{tr } (AB_3) = \frac{35}{8}.$$

Hence, the characteristic polynomial of A is given by (3.1) as

$$a(s) = U_4(s) - 5U_3(s) + \frac{39}{4} U_2(s) - \frac{19}{2} U_1(s) + \frac{35}{8} U_0(s).$$

4.4. Bessel Basis. We consider the family $B_n = B_n^0$. From (3.20), $a_1 = -1 - \text{tr } A = -6$, and from (3.21)

$$B_1 = a_1 I + A = \begin{bmatrix} -5 & -4 & -1 & -4 \\ 2 & -6 & 5 & -4 \\ -1 & 1 & -8 & 3 \\ -1 & 4 & -1 & 0 \end{bmatrix}.$$

Using (3.20), we get

$$a_2 = -\frac{2}{21} - \frac{1}{6} \text{tr } B_1 - \frac{1}{2} \text{tr } (AB_1) = \frac{102}{7},$$

and from (3.21)

$$B_2 = a_2 I + \frac{1}{35} B_0 + AB_1 = \begin{bmatrix} \frac{33}{5} & 3 & -9 & 9 \\ -11 & -\frac{22}{5} & -38 & 7 \\ 6 & 8 & \frac{168}{5} & -6 \\ 8 & 3 & 23 & -\frac{2}{5} \end{bmatrix}.$$

Using again (3.20)

$$a_3 = -\frac{1}{30} \text{tr } B_1 - \frac{1}{6} \text{tr } B_2 - \frac{1}{3} \text{tr } (AB_2) = -\frac{289}{15},$$

and from (3.21)

$$B_3 = a_3 I + \frac{1}{15} B_1 + AB_2 = \frac{1}{3} \begin{bmatrix} -21 & 1 & 52 & -35 \\ 34 & 43 & 175 & -32 \\ -17 & -43 & -141 & 27 \\ -26 & -31 & -116 & 10 \end{bmatrix}.$$

Finally

$$a_4 = -\frac{1}{12} \text{tr } B_2 - \frac{1}{4} \text{tr } B_3 - \frac{1}{4} \text{tr } (AB_3) = \frac{84}{5}.$$

Hence, the characteristic polynomial of A is given by (3.1) as

$$a(s) = B_4(s) - 6B_3(s) + \frac{102}{7} B_2(s) - \frac{289}{15} B_1(s) + \frac{84}{5} B_0(s).$$

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